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Relative Yoneda Cohomology for Operator Spaces*

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We provide two alternate presentations of the completely bounded Hochschild cohomology. One as a relative Yoneda cohomology, i.e., as equivalence classes of n-resolutions which are relatively split, and the second as a derived functor. The first presentation makes clear the importance of certain relative notions of injec- [illegible] earlier by Christensen and Sinclair. We prove that the relatively amenable C^*-algebras are precisely the nuclear C^*-algebras, and hence exactly those which are amenable as Banach algebras. In a similar vein we prove that the only relatively projective C^*-algebras are finite dimensional. This result implies that the only C^*-algebras that are projective as Banach algebras are finite dimensional, a result first obtained by Selivanov and Helemskii.

In a slightly different direction we prove that $B(H)$ viewed as a bimodule over the disk algebra with the left action given by multiplication by a coisometry and the right action given by multiplication by an isometry is an injective module. This result is in some sense a generalization of the Sz.-Nagy-Foias commutant lifting theorem or of the hypoprojectivity introduced by Douglas and the author.

1. INTRODUCTION

Let $B(H)$ denote the bounded linear operators on a Hilbert space H, $A \subseteq B(H)$ be a subalgebra, and let $X \subseteq B(H)$ be a subspace such that $A \cdot X = \{a \cdot x : a \in A, x \in X\} \subseteq X$ and $X \cdot A = \{x \cdot a : a \in A, x \in X\} \subseteq X$. In this case X can be viewed as an A-bimodule and one can introduce the completely bounded Hochschild cohomology, $H^n_{cb}(A; X)$, which is defined in a manner analogous to the usual Hochschild cohomology groups except that the n-linear maps from A into X are required to satisfy a certain extra boundedness condition, called "complete" boundedness. These groups have

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been studied extensively, see for example the text [SS] and play a central role in much current operator algebra theory.

In algebra there are many relations between the Hochschild groups and the two variable ext-functor. Moreover the ext-functor has at least two presentations, one as a derived functor of the hom functor and a second as equivalence classes of resolutions of length n, the Yoneda presentation. This latter presentation makes many results about Hochschild cohomology transparent and brings out more clearly the role played by projective and injective modules.

In this paper we construct versions of the Yoneda cohomology appropriate to the setting of completely bounded maps. We prove some isomorphism theorems between the completely bounded Hochschild cohomology and relative Yoneda cohomology in the sense of [GS]. This presentation allows us to give proofs of some known results about completely bounded Hochschild cohomology that parallel the proofs found in algebra texts.

The fact that completely bounded Hochschild cohomology corresponds to a relative rather than absolute Yoneda cohomology, means that a central role is played by relative notions of injectivity and projectivity. These relative concepts seem to have gone largely unnoticed.

For example, while the (absolutely) injective von Neumann algebras are a fairly small and special class, we prove that every von Neumann algebra is relatively injective (Theorem 5.8) as a bimodule over itself and this fact leads immediately to the conclusion that $H^n_{cb}(M; M) = 0$ for every von Neumann algebra M a result obtained earlier in [CS3].

In a similar vein we are lead to study relative projectivity and relative amenability. For C^*-algebras, we are able to prove that these notions coincide with their Banach algebra counterparts.

The ideas and results in this paper borrow heavily from several sources. Most notably the book of Helemskii [He], where the Banach algebra/Banach module case is worked out in detail and the important role of "relative" theories is apparent, but in a different language. The deepest technical results are modified from [CS3]. Our presentation of relative Yoneda theories is motivated by [GS]. There are several peculiarities of completely bounded module actions which forced this presentation upon us. In the first place, if Y is a completely bounded right B-module, then the natural left action of the opposite algebra on Y need not be completely bounded. This prevents one from reducing $A-B$-bimodules to left $A \otimes B^{op}$-modules. In addition many natural constructions, such as taking operator space duals of completely bounded modules, do not yield new completely bounded modules. Thus many constructions and isomorphisms which occur in algebra and even in the Banach algebra setting must be finessed in this setting.

2. PRELIMINARIES

For clarity of exposition, we assume throughout these notes that all algebras are unital, unless specifically stated otherwise, and that all module actions are unital, i.e., the algebra unit acts as the identity map. Our results extend readily to algebras with a bounded approximate identity provided one restricts attention to "essential" modules, i.e., those for which the approximate identity of the algebra acts as an approximate identity on the module. This is readily seen by the simple device of adjoining a unit to the algebra.

We assume that the reader is familiar with the concepts of operator spaces, operator algebras, completely bounded maps and the Haagerup tensor product. For some key facts see [SS]. We will only review the most salient features.

Given subalgebras $A, B \subseteq B(H)$ and a space of operators $X \subseteq B(H)$ with $A \cdot X \cdot B = \{axb : a \in A, x \in X, b \in B\} \subseteq X$ we can regard X as an $A-B$-bimodule and the trilinear map, $A \times X \times B \to X$ defined by $(a, x, b) \to axb$ is completely contractive in the sense of [CS1]. Conversely, there are axiomatic characterizations of operator spaces and operator algebras and any (abstract) operator space X which is an $A-B$-bimodule over (abstract) unital operator algebras A and B such that the module action is completely contractive has such a representation [CES, BMP]. Recently, Blecher [Bl1, Bl2], has extended this in two directions. First he has shown that any algebra A which is an operator space and such that the product pairing, $A \times A \to A$ is completely bounded is completely boundedly isomorphic to an operator algebra (this requires no unit) and furthermore if a bimodule action $A \times X \times B \to X$ is only completely bounded then A, X, and B have a representation up to completely bounded isomorphism of the above type.

The result that we will need is implied by Blecher's result, but is considerable less general, so we supply an ad hoc proof.

Proposition 2.1. *Let A, B be (unital) operator algebras and let X be an operator space which is an $A-B$-bimodule and such that the pairing $A \times X \times B \to X$ is completely bounded with completely bounded norm c. Then there exists a Hilbert space H a completely contractive linear map $\varphi : X \to B(H)$ and completely contractive homomorphisms $\rho : A \to B(H)$, $\theta : B \to B(H)$ satisfying $\rho(a)\, \varphi(x)\, \theta(b) = \varphi(a \cdot x \cdot b)$ and $\|(x_{ij})\| \leqslant c\, \|(\varphi(x_{ij}))\|$ for all x_{ij} in X.*

Proof. The map $\pi : A \otimes_h X \otimes_h B \to X$ given by $\pi(a \otimes x \otimes b) = axb$ is completely bounded with $\|\pi\|_{cb} = c$. Let $Y = A \otimes_h X \otimes_h B/\ker \pi$ then Y is an operator space. Since the $A-B$-bimodule action $a_1 \cdot (a \otimes x \otimes b) \cdot b_1 = (a_1 a) \otimes x \otimes (bb_1)$ on $A \otimes_h X \otimes_h B$ is completely contractive and $\ker \pi$ is a closed $A-B$-submodule, Y is a completely contractive $A-B$-bimodule. Hence, by applying the representation theorem of [PS] to the trilinear map $A\times Y\times B\to Y, (a, y, b)\to ayb$ and mimicing the proof of [CES], there exists a complete isometry $\gamma: Y\to B(H)$ and ρ and θ satisfying $\rho(a)\,\gamma(y)\,\theta(b)=\gamma(ayb)$. Now set $\varphi(x)=\gamma(q(1\otimes x\otimes 1))$ where $q: A\otimes_h X\otimes_h B\to Y$ denotes the quotient map. ∎

3. RELATIVE YONEDA COHOMOLOGY FOR OPERATOR SPACES

Let A and B be (unital) operator algebras, we let ${}_A O_B$ denote the collection of operator spaces X which are (unital) $A-B$-bimodules with the property that the module pairing $A\times X\times B\to X$ is completely bounded. We call such an X a *completely bounded $A-B$-bimodule*. Given $A-B$-bimodules X and Y which are operator spaces, we let $CB_A(X, Y)_B$ denote the completely bounded maps from X to Y which are $A-B$-bimodule maps.

Let $X=E_0, E_1, ..., E_{n+1}=Y$ be completely bounded $A-B$-bimodules and let $\varphi_i\in CB_A(E_{i-1}, E_i)_B$ be given such that the sequence,

$$\xi: 0\longrightarrow X\xrightarrow{\varphi_1} E_1\longrightarrow\cdots\xrightarrow{\varphi_{n+1}} Y\to 0$$

is exact ($\ker\varphi_i=\operatorname{im}\varphi_{i-1}$), then we call ξ an *n-extension* of X by Y. If

$$\xi': 0\to X\xrightarrow{\varphi'_1} E'_1\longrightarrow\cdots\xrightarrow{\varphi'_{n+1}} Y\longrightarrow 0$$

is another n-extension of X by Y then we write $\xi\to\xi'$ (or $\xi'\leftarrow\xi$) provided that there exist maps, $\psi_i\in CB_A(E_i, E'_i)_B$ with $\psi_0=\mathrm{id}_X$, $\psi_{n+1}=\mathrm{id}_Y$, such that the resulting diagram commutes. We let $\approx$ denote the equivalence relation generated by this partial order. That is, $\xi\approx\xi'$ if and only if there exist n-extensions $\zeta_1, ..., \zeta_{2m}$ of X by Y such that $\xi\to\zeta_1\leftarrow\zeta_2\cdots\leftarrow\zeta_{2m}\to\xi'$.

In [MacL] it is shown that $\xi\approx\xi'$ if and only if there exists ζ such that $\xi\to\zeta\leftarrow\xi'$ and the same proof readily modifies to this situation. We let $[\xi]$ denote the equivalence class of ξ.

If X and Y are completely bounded $A-B$-bimodules, then one readily sees that setting

$$\|(x_{ij}\oplus y_{ij})\|=\max\{\|(x_{ij})\|, \|(y_{ij})\|\}$$

defines an operator space structure on $X\oplus Y$, called the *direct sum*, and that the resulting $A-B$-bimodule is completely bounded. Using this observation one sees that given two n-extensions of X by Y, one can follow the usual construction of the *Baer sum* and one obtains a new n-extension. The salient point being that each of the modules obtained is a completely bounded $A-B$-bimodule. Given the Baer sum, denoted $\xi+\xi'$, one verifies readily that its equivalence class depends only on the equivalence classes of ξ and ξ' and that $\xi+\xi' \approx \xi'+\xi$. Thus, setting $[\xi]+[\xi']=[\xi+\xi']$ defines a commutative binary operation on the collection of equivalence classes of n-extensions of X by Y. One can prove that this is an abelian group with 0 the equivalence class of the resolution

$$0 \to X \to X \to 0 \cdots 0 \to Y \to Y \to 0$$

for $n>1$ and the equivalence class of

$$0 \to X \to X \oplus Y \to Y \to 0$$

when $n=1$.

We will denote this group by $E_{cb}xt^n(A-B;\, Y, X)$ and it will be called the nth *absolute completely bounded Yoneda cohomology group.*

We have been admittedly terse in our definition of this group, but this is because it will play no future role, other than as a comparison and motivator for the relative cohomology groups.

To define the relative theory, we fix a pair of subalgebras $1 \in C \subseteqq A$, $1 \in D \subseteq B$. An n-extension ξ of X by Y is called $C-D$-*split* provided that there exist $\psi_i \in CB_C(E_i, E_{i-1})_D$ such that $\varphi_i \circ \psi_i \circ \varphi_i = \varphi_i$ for all i. This is equivalent to requiring that E_i is cb-isomorphic as an operator space to $\ker \varphi_{i+1} \oplus \ker \varphi_{i+2}$ and, moreover this is a $C-D$-bimodule isomorphism when $\ker \varphi_{i+1} \oplus \ker \varphi_{i+2}$ is given the trivial *diagonal* action $c \cdot (w \oplus z) \cdot d = (cwd) \oplus (czd)$.

Thus, every $C-D$-split n-extension is equivalent to one of the form

$$\xi : 0 \to X \to X \oplus K_1 \to K_1 \oplus K_2 \to \cdots \to K_{n-1} \oplus Y \to Y \to 0$$

where the operator space structures are all the direct sum, $K_0 = X$, $K_1, \ldots, K_{n-1}$, $Y = K_n$ and $K_i \oplus K_{i+1}$ are all completely bounded $A-B$-bimodules and the module actions restricted to $C-D$ are the trivial diagonal actions. Moreover the horizontal maps are all of the form $k_i \oplus k_{i+1} \to k_{i+1} \oplus 0$. The key point is that the $A-B$-bimodule actions are not necessarily the diagonal actions.

It is worthwhile to examine the case of a $C-D$-split 1-extension of X by Y. This is a short exact sequence

$$\xi : 0 \to X \to X \oplus Y \to Y \to 0$$

where the maps are the inclusion and quotient maps, respectively. For the inclusion and quotient maps to be $A-B$-bimodule maps we need the module actions on $X\oplus Y$ to satisfy,

$$a\cdot(x\oplus y)\cdot b=(axb+\gamma(a,\ y,\ b))\oplus ayb$$

where $\gamma\colon A\times Y\times B\to X$ is trilinear. Since the module actions on X and Y are completely bounded the new action will be completely bounded if and only if γ is completely bounded. Associativity of the action $(a_1\cdot a_0)(x\oplus y)(b_0\cdot b_1)=a_1\cdot(a_0\cdot(x\oplus y)\cdot b_1)$ translates into the *$A-B$-derivation* equation:

$$\gamma(a_1a_0,\ y,\ b_0b_1)=a_1\gamma(a_0,\ y,\ b_0)\,b_1+\gamma(a_1,\ a_0yb_0,\ b_1). \tag{1}$$

The fact that the action is diagonal as a $C-D$-bimodule action is equivalent to the fact that $\gamma(c,\ y,\ d)=0$ for all $c\in C$, $d\in D$ and $y\in Y$. Combined with (1) this fact is equivalent to:

$$\gamma(c_1ac_2,\ y,\ d_2bd_1)=c_1\gamma(a,\ c_2yd_2,\ b)\,d_1 \tag{2}$$

for all $a\in A$, $b\in B$, $c_1, c_2\in C$, $d_1, d_2\in D$ and $y\in Y$, which we call *$C-D$-multimodularity*.

Such a map γ will be called a *completely bounded* $(A-B,\ C-D)$*-derivation.*

Conversely, given such a γ, then defining $a\cdot(x\oplus y)\cdot b=(axb+\gamma(a,\ y,\ b))\oplus ayb$ makes the operator space $X\oplus Y$ into a completely bounded $A-B$-bimodule which we denote $X\oplus_\gamma Y$, and the sequence

$$\xi\colon 0\to X\to X\oplus_\gamma Y\to Y\to 0$$

is $C-D$-split. When γ is the 0-map then the $A-B$-bimodule action is diagonal.

Remark 3.1. $A-B$-derivations do not occur much in the algebra literature since algebraically an (A, B)-bimodule is just a left $A\otimes B^{op}$-module and an $A-B$-derivation is an $A\otimes B^{op}$-derivation. However, we are faced with a dilemma. There is no tensor norm which simultaneously makes $A\otimes B^{op}$ an operator algebra and a completely bounded $A-B$-bimodule a completely bounded left $A\otimes B^{op}$-module. This is essentially because of the facts that the Haagerup norm is noncommutative and $A\otimes_h B^{op}$ is generally not an operator algebra. This difficulty does not arise in the Banach algebra literature since one can just take the Banach space projective tensor product. For these reasons it is essential that we keep track of both algebras.

It is readily checked that a 1-extension ξ is equivalent to ξ' if and only if there exists a completely bounded $A-B$-bimodule map $R\colon X\oplus_\gamma Y\to X\oplus_{\gamma'} Y$ which restricts to the identity on X and quotients to the identity on Y. Such a map R must have the form $R(x \oplus y) = (x + T(y)) \oplus y$ for some T in $CB(Y, X)$. The fact that R is an $A - B$-bimodule map reduces to the identity

$$\gamma(a, y, b) = aT(y)\, b - T(ayb) + \gamma'(a, y, b). \tag{3}$$

Since $\gamma'(c, y, d) = \gamma(c, y, d) = 0$ for all $c \in d, d \in D$, we have that T is actually a $C - D$-bimodule map. Consequently, we define a completely bounded $(A - B, C - D)$-derivation γ to be *inner* if and only if there exists a completely bounded $C - D$-bimodule map from Y to X satisfying

$$\gamma(a, y, b) = aT(y)\, b - T(ayb). \tag{4}$$

We denote the inner derivation defined by (4) by δ_T. Thus, ξ and ξ' are equivalent if and only if $\gamma - \gamma'$ is inner.

Extending the above, we have that every $C - D$-split n-extension of X by Y is equivalent to one of the form

$$\xi: 0 \to X \to X \oplus_{\gamma_1} K_1 \to \cdots K_{n-1} \oplus_{\gamma_n} Y \to Y \to 0$$

where we set $X = K_0$, $Y = K_n$ and each $\gamma_i: A \times K_i \times B \to K_{i-1}$ is a completely bounded $(A - B, C - D)$-derivation.

In this setting it is elementary to define push-outs and pull-backs. Given X, Y, Z in ${}_AO_B$, $S \in CB_A(Z, Y)_B$ and $\gamma: A \times Y \times B \to X$ a completely bounded $(A - B, C - D)$-derivation, let $\delta: A \times Z \times B \to X$ be defined by $\delta(a, z, b) = \gamma(a, S(z), b)$. The following diagram then commutes:

$$\begin{tikzcd}
\xi': 0 \arrow[r] & X \arrow[r] \arrow[d, "1"'] & X \oplus_\delta Z \arrow[r] \arrow[d, "1 \oplus S"] & Z \arrow[r] \arrow[d, "S"] & 0 \\
\xi: 0 \arrow[r] & X \arrow[r] & X \oplus_\gamma Y \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

and we denote $\xi' = \xi S$, $\delta = \gamma S$ and ξ' is called the *pull-back* of ξ along S.

Similarly, given $R \in CB_A(X, Z)_B$ we get a commuting diagram

$$\begin{tikzcd}
\xi': 0 \arrow[r] & X \arrow[r] \arrow[d, "R"'] & X \oplus_\gamma Y \arrow[r] \arrow[d, "R \oplus 1"'] & Y \arrow[r] \arrow[d, "1"'] & 0 \\
\xi'': 0 \arrow[r] & W \arrow[r] & W \oplus_\beta Y \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

by setting $\beta: A \times Y \times B \to Y$, $\beta(a, y, b) = R(\gamma(a, y, b))$ and we denote $\xi'' = R\xi$, $\beta = R\gamma$ and ξ'' is called the *push-out* of ξ by R. Using the Yoneda composition of resolutions [MacL] one readily sees how to define pull-backs and push-outs of $C - D$-split n-extensions.

Given $C-D$-split n-extensions,

$$\xi : 0 \longrightarrow E_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_n} E_n \longrightarrow 0$$

$$\xi : 0 \longrightarrow F_0 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_n} E_n \longrightarrow 0$$

set

$$\xi \oplus \zeta : 0 \longrightarrow E_0 \oplus F_0 \xrightarrow{\varphi_1 \oplus \psi_1,} \cdots \xrightarrow{\varphi_n \oplus \psi_n} E_n \oplus F_n \longrightarrow 0$$

If ξ, ζ are $C-D$-split n-extensions of X by Y we define $\Delta_Y : Y \to Y \oplus Y$ by $\Delta(y) = y \oplus y$, $\nabla_X : X \oplus X \to X$ by $\nabla_X(x_1, x_2) = x_1 + x_2$. Given $C-D$-split n-extensions ξ and ξ' of X by Y we define their *Baer sum* to be the $C-D$-split n-extension, $\xi + \xi' = \nabla_X((\xi \oplus \xi')\, \Delta_Y)$.

The proof of the following is as in [MacL, III.5] or [GS, Section 12]. We let $E_{cb}xt^n(A-B, C-D; Y, X)$ denote the set of equivalence classes of $C-D$-split n-extensions of X by Y.

THEOREM 3.2. *Let $1 \in C \subseteq A$, $1 \in D \subseteq B$ be operator algebras, X, Y in ${}_A O_B$. Then the Baer sum preserves equivalence classes of $C-D$-split n-extensions of X by Y and $E_{cb}xt^n(A-B, C-D; Y, X)$ is a commutative group under Baer sum. Moreover, if a $C-D$-split n-extension ξ is identified with the n-tuple $(\gamma_1, ..., \gamma_n)$ of $(A-B, C-D)$-derivations, then the inverse of ξ is equivalent to the $C-D$-split n-extension identified with $(-\gamma_1, \gamma_2, ..., \gamma_n)$.*

One sees immediately from (3) that the group $E_{cb}xt^1(A-B, C-D; Y, X)$ is just the space of $(A-B, C-D)$-derivations modulo the inner derivations.

If an n-extension of X by Y is $A-B$-split at any point then it can be shown to be equivalent to the 0 element. This follows as in [MacL, III.5] by introducing the “elementary replacements.” For our purposes it is useful to see how each of these elementary equivalences behaves on derivations. Let

$$\xi : 0 \to K_0 \to K_0 \oplus_{\gamma_1} K_1 \to \cdots \to K_{n-1} \oplus_{\gamma_n} K_n \to K_n \to 0$$

be an n-extension, and let

$$\xi_i : 0 \to K_{i-1} \to K_{i-1} \oplus_{\gamma_i} K_i \to K_i \to 0$$

so that ξ is the Yoneda composition [MacL], $\xi_1 \circ \cdots \circ \xi_n$.

Now let $\xi = \xi_1 \circ \cdots \circ \xi_n$, $\xi' = \xi'_1 \circ \cdots \xi'_n$ be two n-extensions of X by Y written as Yoneda compositions, so that $X = K_0 = K'_0$, $Y = K_n = K'_n$ and let $(\gamma_1, ..., \gamma_n)$, $(\gamma'_1, ..., \gamma'_n)$ be the defining $(A-B, C-D)$-derivations. We define the *elementary replacements* by the following:

(I) ξ' is a *type I replacement* of ξ provided for some i, we have that $\xi_j = \xi'_j$ for $j \neq i$, and ξ_i is equivalent to ξ'_i. That is, there exists $T \in CB_C(K_{i+1}, K_i)_D$ satisfying (3).

(II) ξ' is a *type II replacement* of ξ provided for some i we have that $\xi_j = \xi'_j$ for $j \neq i, i+1$ while $\xi_i = \xi'_i R$, $\xi'_{i+1} = R\xi_{i+1}$ for some $R \in CB_A(K'_i, K_i)_B$.

(III) ξ' is a *type III replacement* of ξ provided for some i, we have that $\xi_j = \xi'_j$ for $j \neq i, i+1$ while $\xi_{i+1} = S\xi'_{i+1}$, $\xi'_i = \xi_i S$ for some $S \in CB_A(K'_i, K_i)_B$.

Note that II and III are really the same but with the roles of ξ and ξ' reversed. The reason for this is that II defines a morphism of extensions $\xi \to \xi'$ while III defines a morphism $\xi \leftarrow \xi'$. A type I replacement replaces the derivation γ_i by a $\gamma_i - \delta_T$. Moreover, if ξ' is obtained from ξ by replacement then ξ and ξ' are equivalent.

It is worthwhile to examine the general morphism $\alpha: \xi \to \xi'$ between two short exact sequences of $C-D$-split $A-B$-bimodules. Let $\xi: 0 \to X \to X \oplus_\gamma Y \to Y \to 0$, $\xi': 0 \to W \to W \oplus_\mu Z \to Z \to 0$, then it is easily checked that there exists

$$R \in CB_A(X, W)_B, \qquad T \in CB_C(Y, W)_D \qquad \text{and}$$
$$S \in CB_A(Y, Z)_B$$

so that we have α is implemented by the maps $R: X \to W$, $\begin{pmatrix} R & T \\ 0 & S \end{pmatrix}: X \oplus Y \to W \oplus Z$, $S: Y \to Z$ where $\begin{pmatrix} R & T \\ 0 & S \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Rx + Ty \\ Sy \end{pmatrix}$ in operator notation. Factoring

$$\begin{pmatrix} R & T \\ 0 & S \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$$

actually factors α as a product of a push-out, equivalence and a pull-back as follows:

$$\begin{tikzcd}
\xi': 0 \arrow[r] & X \arrow[r] \arrow[d, "R"] & X \oplus_\gamma Y \arrow[r] \arrow[d, "R \oplus I"] & Y \arrow[r] \arrow[d, "I"] & 0 \\
R\xi: 0 \arrow[r] & W \arrow[r] \arrow[d, "I"] & W \oplus_{R\gamma} Y \arrow[r] \arrow[d, "\begin{pmatrix} I & T \\ 0 & I \end{pmatrix}"] & Y \arrow[r] \arrow[d, "I"] & 0 \\
\xi'': 0 \arrow[r] & W \arrow[r] \arrow[d, "I"] & W \oplus_\beta Y \arrow[r] \arrow[d, "I \oplus S"] & Y \arrow[r] \arrow[d, "S"] & 0 \\
\xi': 0 \arrow[r] & W \arrow[r] & W \oplus_\mu Z \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

and we necessarily have the identities $\beta = R\gamma - \delta_T = \mu S$, $\xi'' = \xi' S$ and deduce that $R\xi$ and $\xi' S$ are equivalent extensions of W by Y. These facts are restatements of [MacL, III.1.5 and III.1.8].

The following is the analogue of [MacL, III.5.2.].

THEOREM 3.3. *Let X, Y be in ${}_A O_B$ and let ξ and ξ' be two $C-D$-split n-extensions of X by Y. Then ξ and ξ' are equivalent if and only if ξ' can be obtained from ξ by a finite sequence of elementary replacements.*

Proof. It suffices to show that if $\xi \to \xi'$ then ξ' can be obtained from ξ by a sequence of elementary replacements. This follows by applying the above factorization, repeatedly.

Remark 3.4. In the work of [Bl2] and [Ru2] matrix-normed algebras for which the product map $A \times A \to A$ is only "jointly completely bounded" in the sense of [BP] or equivalently only extends to be completely bounded on the operator space projective tensor product are considered. These include operator algebras as well as some other important algebras such as the Fourier–Stieltjes algebras of groups. Moreover, in this setting it is natural to study operator spaces which are $A-B$-bimodules but which satisfy the weaker condition that the module maps $A \times X \times B \to X$ are only jointly completely bounded. Imitating the above one can construct a relative Yoneda theory, which we denote by $E_{jcb}xt^n(A-B, C-D; Y, X)$. In this theory derivations γ: $A \times Y \times B \to X$ need only be jointly completely bounded and this is equivalent to the induced map, Γ: $A \times B \to CB(Y, X)$ being jointly completely bounded. Moreover, if A and B are algebras in this larger category then $A \widehat{\otimes} B^{op}$ endowed with the operator space projective norm is again in this category and an operator space X has a jointly completely bounded $A-B$-bimodule action if and only if X is jointly completely bounded as a left $A \widehat{\otimes} B^{op}$-module. Thus, further simplifications ensue. However, what is lost is that there is no representation theory or correspondence to concrete operator spaces and algebras.

4. HOCHSCHILD COHOMOLOGY AND THE BAR RESOLUTION

In this section we introduce the appropriate bar resolution in our setting and give the derived functor realization of our relative Yoneda cohomology. We then use this result to prove isomorphism of our Yoneda theory with the relevant completely bounded Hochschild cohomology.

Given Y in ${}_A O_B$ with $C \subseteq A$, $D \subseteq B$ we wish to define a bar resolution as follows. Let $P_0 = A \otimes_{h, C} Y \otimes_{h, D} B$ where h-denotes the Haagerup tensor product and the subscripts C and D indicate that we are quotienting out the C and D actions. These operations are associative, i.e.,

$(A \otimes_{h,C} Y) \otimes_{h,D} B = A \otimes_{h,C} (Y \otimes_{h,D} B)$ completely isometrically and one obtains the same operator space by either forming the completed tensor product $A \otimes_h Y \otimes_h B$ and dividing out the closed subspace generated by tensors of the form $ac \oplus y \oplus db - a \otimes cyd \otimes b$ or by first forming the algebraic tensor product $A \otimes_C Y \otimes_D B$ and endowing it with a Haagerup seminorm. Moreover, it satisfies $\psi: A \times Y \times B \to Z$ is completely bounded and $\psi(ac, y, db) = \psi(a, cyd, b)$ if and only if ψ extends uniquely to a completely bounded linear map $\psi: P_0 \to Z$ with the same completely bounded norm. See [BMP] for a proof of these claims.

Moreover, P_0 is in ${}_A O_B$ with bimodule actions defined by $a_1(a \otimes y \otimes b)\, b_1 = (a_1 a) \otimes y \otimes (bb_1)$ and these are completely contractive actions.

We now define the $(A-B, C-D)$-*bar resolution* of Y by inductively setting $P_{n+1} = A \otimes_{h,C} P_n \otimes_{h,D} B$ and defining $\pi_0 : P_0 \to Y$ by $\pi_0(a \otimes y \otimes b) = ayb$ and $\pi_{n+1} : P_{n+1} \to P_n$ via

$$\begin{aligned}
&\pi_{n+1}(a_{n+1} \otimes \cdots \otimes a_1 \otimes y \otimes b_0 \otimes \cdots \otimes b_{n+1}) \\
&\quad = \sum_{i=1}^{n+1} (-1)^i\, a_{n+1} \otimes \cdots \otimes a_{n+1} \otimes (a_i a_{i-1}) \otimes \cdots \otimes a_0 \otimes y \otimes b_0 \otimes \cdots \\
&\qquad \otimes (b_{i-1} b_i) \otimes \cdots \otimes b_{n+1} \\
&\qquad + a_{n+1} \otimes \cdots \otimes a_1 \otimes a_0 y b_0 \otimes b_1 \otimes \cdots \otimes b_{n+1}.
\end{aligned}$$

For notational reasons we set $Y = P_{-1}$. These maps are readily seen to be completely bounded because π_{n+1} is the sum of $n+2$ complete contractions and hence $\|\pi_{n+1}\|_{cb} \leqslant n+2$.

The sequence

$$\cdots \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} Y \longrightarrow 0 \tag{4.1}$$

is exact. To see this one uses that $\pi_{n+1} \circ \pi_n = 0$ and that defining $\theta_n(q) = 1_A \otimes q \otimes 1_B$ extends to give a well-defined completely contractive map $\theta_n : P_{n-1} \to P_n$ satisfying $\pi_n(\theta_{n+1}(k)) = k$ for k in $\ker \pi_n$. Moreover, since each θ_n is a $C-D$-bimodule map we have that the $(A-B, C-D)$-bar resolution (4.1) is $C-D$-split.

Now fix any X in ${}_A O_B$ and consider the chain complex,

$$\mathscr{C}_{Y,X} : CB_A(P_0, X)_B \xrightarrow{\pi_1^*} CB_A(P_1, X)_B \longrightarrow \cdots,$$

where $\pi_n^*(\psi) = \psi \circ \pi_n$.

We claim that the nth homology group of this chain complex $H^n(\mathscr{C}_{Y,X}) \equiv \ker \pi_{n+1}^* / \operatorname{im} \pi_n^*$ is isomorphic to $E_{cb}xt^n(A-B, C-D; Y, X)$. This will be shown in Theorem 4.

Before proceeding further it is convenient to recall the correspondence between this chain complex and the Hochschild complex.

Note that if $\Psi \in CB_A(P_n, X)_B$ then Ψ is uniquely determined by its values on elements of the form $1 \otimes q \otimes 1, q \in P_{n-1}$. Thus, the map $\theta_n^* : CB_A(P_n, X)_B \to CB(P_{n-1}, X)$ is one-to-one. Its range is easily seen to be in $CB_C(P_{n-1}, X)_D$ since $\theta^*\Psi(c \cdot q \cdot d) = \Psi(1 \otimes cqd \otimes 1) = \psi(c \otimes q \otimes d) = c\theta_n^* \Psi(q)\, d$. Conversely, if $\Phi \in CB_C(P_{n-1}, X)_D$ then the multi-linear map $\psi(a_n, ..., a_0, y, b_0, ..., b_n) = a_n \Phi(a_{n-1} \otimes \cdots \otimes a_0 \otimes y \otimes b_0 \otimes \cdots \otimes b_{n-1})\, b_n$ is easily seen to be completely bounded, $C-D$-multimodular and hence extends to a completely bounded linear map $\Psi \in CB_A (P_n, X)_B$ satisfying $\theta_n^* \Psi = \Theta$. Thus, $\theta_n^* : CB_A(P_n, X)_B \to CB_C(P_{n-1}, X)_D$, $n \geqslant 0$ is an isomorphism, and consequently there are induced maps,

$$\partial_n : CB_C(P_{n-1}, X)_D \to CB_C(P_n, X)_D$$

satisfying $\theta_n^* \pi_n^* = \partial_{n-1}^* \theta_{n-1}^*, n \geqslant 1$. The maps ∂_n are defined as follows,

$$(\partial_1 T)(a \otimes y \otimes b) = aT(y)\, b - T(ayb)$$

and in general

$$\begin{aligned}
&(\partial_n \Phi)(a_n \otimes \cdots \otimes a_0 \otimes y \otimes b_0 \otimes \cdots \otimes b_n)\\
&\quad = a_n \Phi((a_{n-1} \otimes \cdots \otimes a_0 \otimes y \otimes b_0 \cdots \otimes b_{n-1})\, b_n\\
&\qquad - \Phi((a_n a_{n-1}) \otimes \cdots \otimes a_0 \otimes y b_0 \otimes \cdots \otimes (b_{n-1} b_n)) + \cdots\\
&\qquad + (-1)^n\, \Phi(a_n \otimes \cdots \otimes a_1 \otimes a_0 y b_0 \otimes b_1 \otimes \cdots b_n).
\end{aligned}$$

Definition 4.1. Let $1 \in C \subseteq A$, $1 \in D \subseteq B$ be operator algebras and let X, Y be in ${}_A O_B$. A completely bounded $(2n+1)$-multilinear map $\psi : A \times \cdots \times A \times Y \times B \times \cdots \times B \to X$ which is $C-D$-multimodular is called an *n-cocycle* provided $\partial_n \Psi = 0$ where Ψ denotes the linearization of ψ to P_{n-1} and we denote the set of such maps by $Z_{cb}^n(A-B, C-D; Y, X)$. It is called an *n-coboundary* if $\Psi = \partial_{n-1} \Phi$ for some $(2n-1)$-multilinear completely bounded $C-D$-multimodular map $\varphi : A \times \cdots \times A \times Y \times B \times \cdots \times B \to X$ and we denote this set by $B_{cb}^n(A-B, C-D; Y, X)$. We define the nth *completely bounded relative Hochschild group* to be.

$$\begin{aligned}
&H_{cb}^n(A-B, C-D; Y, X)\\
&\quad = Z_{cb}^n(A-B, C-D; Y, X)/B_{cb}^n(A-B, C-D; Y, X).
\end{aligned}$$

The calculations above are summarized by the following result.

Proposition 4.2. *There is an isomorphism,* $H_{cb}^n(A-B, C-D; Y, X) \cong H^n(\mathscr{C}_{Y, X})$.

Proof. $H^n_{cb}(A-B, C-D; Y, X)$ is clearly the homology of the complex $\{CB_C(P_{n-1}, X)_D, \partial^*_{n-1}\}_{n\geqslant 0}$ which is chain isomorphic to the complex $\mathscr{C}_{Y,X}$. ∎

To see how these relate to the usual completely bounded Hochschild cohomology for X in ${}_A O_A$ as defined in say [SS], we have the following result.

PROPOSITION 4.3. *Let A be a unital operator algebra and let X be in ${}_A O_A$, then*

$$H^n_{cb}(A; X), H^n_{cb}(A-A, \mathbb{C}-A; A, X),$$

and $H^n_{cb}(A-A, A-\mathbb{C}; A, X)$ are isomorphic.

Proof. Let $\psi \in Z^n_{cb}(A-A, \mathbb{C}-A; A, X)$, so that ψ is $(2n+1)$-linear, completely bounded and $\psi: A\times \cdots \times A \to X$. Note that since ψ is $\mathbb{C}-A$-multimodular,

$$\psi(a_{n-1}, ..., a_0, a, b_0, ..., b_{n_1}) = \psi(a_{n-1}, ..., a_0, 1, 1, ...1)\, ab_0 \cdots b_{n-1}.$$

Let δ be the n-linear map defined by $\delta(a_{n-1}, ..., a_0) = \psi(a_{n-1}, ..., a_0, 1, ..., 1)$, so that $\delta: A\times \cdots \times A \to X$ is completely bounded. Clearly, every such δ defines a corresponding ψ. Now check that $\partial_n \psi = 0$ if and only if $\delta \in Z^n_{cb}(A; X)$ as defined in the usual Hochschild cohomology, and that $\psi \in B^n_{cb}(A-A, \mathbb{C}-A; A, X)$ if and only if $\delta \in B^n_{cb}(A; X)$. This proves the first isomorphism.

The isomorphism of $H^n_{cb}(A; X)$ and $H^n_{cb}(A-A, A-\mathbb{C}; A, X)$ follows similarly. ∎

The following result can essentially be deduced from [MacL]. Our proof is perhaps different.

THEOREM 4.4. *Let $1 \in C \subseteq A$, $1 \in D \subseteq B$ be operator algebras and let X, Y be in ${}_A O_A$ then $E_{cb}xt^n(A-B, C-D; Y, X)$ is isomorphic to $H^n_{cb}(A-B, C-D; Y, X)$ for all $n \geqslant 1$.*

Proof. We record the map and its inverse. First assume that we are given,

$$\xi: 0 \to X \to X \oplus_{\gamma_1} K_1 \to \cdots K_{n-1} \oplus_{\gamma_n} Y \to Y \to 0.$$

Define a $(2n+1)$-linear map $\psi: A\times \cdots \times A \times Y \times Y \times B \times \cdots \times B \to X$ via $\psi = \gamma_1 \circ \cdots \gamma_n$, that is, inductively define $\psi_0 = \gamma_n$,

$$\psi_1(a_1, a_0, y, b_0, b_1) = \gamma_{n-1}(a_1, \gamma_n(a_0, y, b_0), b_1), \qquad \text{and}$$

$$\psi_k(a_k, ..., a_0, y, b_0, ..., b_k)$$
$$= \gamma_{n-k}(a_k, \psi_{n-1}(a_{n-1}, ..., a_0, y, b_0, ..., b_{n-1}), b_k)$$

with $\psi = \psi_{n-1}$.

The fact that each γ_k is completely bounded and $C-D$-multimodular implies that ψ is also. Next one checks that $\partial_n \psi = 0$, this follows readily because ψ is a "product" of $A \otimes B^{op}$-derivations when one regards each γ_k as a derivation from $A \otimes B^{op}$ into the $A \otimes B^{op}$-module of maps from K_k to K_{n-1}. Thus, $\psi \in Z^n_{cb}(A-B, C-D; Y, X)$. Denoting this element by $\psi(\xi)$, one checks that

$$\psi(\xi_1 + \xi_2) = \psi(\xi_1) + \psi(\xi_2)$$

and that $\psi(\xi_0) = 0$.

Next one considers the elementary replacements. If one alters ξ by a type II replacement then γ_i' and $\gamma_{i+1}' = R\gamma_i$ in the definition of $\psi(\xi')$. Hence, $\psi(\xi) = \psi(\xi')$ as functions. Similarly, a type III replacement does not change the value of $\psi(\xi)$ as a function. A type I replacement substitutes γ_i in the definition of $\psi(\xi)$ by $\gamma_i + \delta_T$ in the definition of $\psi(\xi')$. Hence, $\psi(\xi') - \psi(\xi)$ is a product of derivations, one of which is inner and this can be shown to belong to $B^n_{cb}(A-B, C-D; Y, X)$.

Since the equivalence relation on $E_{cb}xt^n(A-B, C-D; Y, X)$ is generated by the elementary replacements, we have that the map $\xi \to \psi(\xi)$ induces a group homomorphism from $E_{cb}xt^n(A-B, C-D; Y, X)$ to $H^n_{cb}(A-B, C-D; Y, X)$.

One now constructs an inverse map as follows. Given $\psi \in Z^n_{cb}(A-B, C-D; Y, X)$ there exists $\Phi \in CB_A(P_n, X)_B$ with $\theta_n^* \Phi = \Psi$ and $\pi_{n-1} \Phi = \Phi \circ \pi_{n-1} = 0$. Since range $\pi_{n-1} = \ker \pi_n$ there is a well-defined completely bounded map $\dot{\Phi}: P_n/\ker \pi_n \to X$ which is an $A-B$-bimodule map. But recall that $P_n/\ker \pi_n \cong \ker \pi_{n-1}$ as operator spaces and $A-B$-bimodules. Hence setting

$$\mathscr{P}_{n-1}: 0 \to \ker \pi_{n-1} \to P_{n-1} \to \cdots \to P_0 \to Y \to 0,$$

we may push-out $\mathscr{P}_{n-1}$ to obtain an n-resolution of X by Y, $\dot{\Phi}\mathscr{P}_{n-1}$.

One verifies that if $\phi = \psi(\xi)$ for some ξ and $\theta_n^* \Phi = \Psi$, then $\psi(\dot{\Phi}\mathscr{P}_{n-1}) = \phi$.

By the isomorphism of $H^n_{cb}(A-B, C-D; Y, X)$ with $H^n(\mathscr{C}_{Y,X})$, to finish the proof, it is enough to show that when $\Phi = \pi_n^* \Gamma$ for some

$\Gamma \in CB_A(P_{n-1}, X)_B$ then $\dot{\Phi}\mathscr{P}_{n-1}$ is in the equivalence class of the trivial extension. Note that the map Γ makes the square,

$$\begin{tikzcd} 0 \arrow[r] & \ker \pi_{n-1} \arrow[r] \arrow[d, "\dot{\Phi}"'] & P_{n-1} \arrow[d, "\Gamma"] \\ & X \arrow[r] & X \end{tikzcd}$$

commute.

Thus, it is enough to show that if

$$\begin{tikzcd} \xi : 0 \arrow[r] & E \arrow[r] \arrow[d, "\varphi"'] & E \oplus_\gamma F \arrow[r] \arrow[d, "\beta"] & F \arrow[r] & 0 \\ & X \arrow[r] & X & & \end{tikzcd}$$

commutes, then the push-out $\phi\xi$ splits.

Recall $\phi\xi : 0 \to X \to X \oplus_{\phi\gamma} F \to F \to 0$, where $\phi\gamma(a, f, b) = \phi(\gamma(a, f, b))$. Write $\beta(\mathrm{e} \oplus f) = \phi(e) + \alpha(f)$, then since these are module maps we have

$$a\alpha f(b) = a\beta(0 \oplus f)\, b = \beta(a(0 \oplus f)\, b) =$$

$$\beta(\gamma(a, f, b) \oplus afb) = \phi\gamma(a, f, b) + \alpha(afb)$$

and so $\phi\gamma$ is the inner derivation generated by α and hence yields a trivial extension.

This completes the proof of the theorem. ∎

Corollary 4.5. *Let A be a unital operator algebra and let X be in ${}_AO_A$ then*

$$H^n_{cb}(A; X), \quad E_{cb}xt^n(A - A, \mathbb{C} - A; A, X),$$

and $E_{cb}xt^n(A - A, A - \mathbb{C}; A, X)$ are isomorphic.

If one starts with $\phi \in H^n_{cb}(A; X)$ and follows the steps of the above proof then one obtains a "canonical" n-cocycle ϕ' which differs from ϕ by a coboundary and has the property that it factors as a "product" of derivations, albeit into different A-bimodules.

We close this section by focusing on the case of left A-modules. This material is not essential to later sections. Since every left A-module is automatically a right $\mathbb{C}$-module, this theory is covered by the general theory by taking $B = D = \mathbb{C}$. Given $C \subseteq A$, note that $P_n = A \otimes_{h, C} P_{n-1} \otimes_{h, \mathbb{C}} \mathbb{C} = A \otimes_{h, C} P_{n-1}$ which corresponds to identifying $\gamma : A \times Y \times \mathbb{C} \to X$ with $\delta : A \times Y \to X$ via $\delta(a, y) = \gamma(a, y, 1)$. In the Hochschild-version of this theory n-cocyles are completely bounded maps

$\psi: A \times \cdots \times A \times Y \to X$. It is tempting to identify this with a map $L: A \times \cdots \times A \to CB(Y, X)$, however, the norms do not agree. This is because the identification of $\psi: W \times Y \to X$ with $L: W \to CB(Y, X)$ corresponds to the operator-space projective norm [BP]. That is, $\|L\|_{cb} = \|\psi\|_{jcb} = \|\Psi\|_{cb}$ where $\Psi: W \widehat{\otimes} Y \to X$ denotes the linear extension of ψ to the operator space projective tensor product. One can work out a parallel Yoneda and Hochschild theory for the operator space projective tensor product as in Remark 3.4 and we expand upon this theme later.

One can identify completely bounded maps $\psi: W \times Y \to X$ with completely bounded maps $L: W \to CB(Y, X)$ by the simple device of giving $CB(Y, X)$ an "unusual" matrix-norm structure. This was first pointed out in [Pa2] and [BP]. we now describe this in some detail.

For $T = (T_{ij})$ an $n \times m$ matrix over $CB(Y, X)$, regard T as a map $T^{(r)}: M_{m,r}(Y) \to M_{n,r}(X)$ via matrix-multiplication, $T^{(r)}((y_{k,j})) = (\sum_{k=1}^{m} T_{ik}(y_{kj}))$ and define $\||T|\| = \sup_r \|T^{(r)}\|_{cb}$.

It is not hard to verify that with this structure $CB(Y, X)$ becomes a matrix-normed space, i.e., the norm on $M_{n,m}(CB(Y, X))$ makes it a contractive $M_n - M_m$-bimodule, which is generally not an operator space. We use $CB^{\ell}(Y, X)$ to denote $CB(Y, X)$ supplied with this matrix-norm structure. It is straight-forward to verify that if we identify $\psi: W \times Y \to X$ with $L: W \to CB^{\ell}(Y, X)$ then $\|\psi\|_{cb} = \|L\|_{cb}$.

Proposition 4.6. *Let X, Y be in ${}_A O_{\mathbb{C}}$. Then $CB^{\ell}(Y, X)$ endowed with the module actions $(a \cdot T)(y) = a \cdot (T(y))$, $(T \cdot a)(y) = T(ay)$ is a completely bounded $A - A$-bimodule that is, there exists a constant C such that*

$$\left\|\left|\left(\sum_{k,l} a_{ik} \cdot T_{kl} \cdot b_{lj} \right)\right|\right\| \leqslant C \, \|(a_{ij})\| \; \||(T_{kl})|\| \; \|(b_{kj})\|$$

for any matrices (a_{ij}), $(b_{k,j})$ in $M_n(A)$, and (T_{kl}) in $CB(Y, X)$. Moreover, for any matrix of cb-maps, $\|(T_{ij})\|_{cb} \leqslant \||(T_{i,j})|\|$.

Proof. The first statement is straightforward and the second statement follows from the fact that the Haagerup tensor norm is dominated by the operator space projective. ∎

Now for any matrix-normed space $\mathbb{Z}$ that is an A-bimodule one can define $H^n_{cb}(A; \mathbb{Z})$ as the Hochschild cohomology one obtains by insisting that all n-cocycles are completely bounded.

Proposition 4.7. *Let A be an operator algebra, X, Y in ${}_A O_{\mathbb{C}}$. Then, $E_{cb}xt^n(A - \mathbb{C}, \mathbb{C} - \mathbb{C}; Y, X)$ is isomorphic to $H^n_{cb}(A; CB^{\ell}(Y, X))$.*

Proof. This is a direct application of the earlier results after noting that $\psi: A \times \cdots \times A \times Y \to X$ is completely bounded if and only if the associated map $L: A \times \cdots \times A \to CB^{\ell}(Y)X)$ is completely bounded. ∎

Now let H, be a Hilbert space and recall that a unital homomorphism $\rho: A \to B(H)$ makes H a left A-module. In [BMP] it is proved that ρ is completely bounded if and only if H endowed with its' column operator space structure H_c is in ${}_A O_{\mathbb{C}}$.

This leads to the following result.

PROPOSITION 4.8. *Let H, K be Hilbert spaces and let $\rho: A \to B(H)$, $\theta: A \to B(K)$ be unital completely bounded homomorphisms, then $B(K, H) = CB(K_c, H_c) = CB^{\ell}(K_c, H_c)$ completely isometrically as matrix-normed spaces. Consequently, $H^n_{cb}(A; B(K, H))$ is isomorphic to $E_{cb}xt^n(A - \mathbb{C}, \mathbb{C} - \mathbb{C}; K_c, H_c)$.*

More generally we have the following result.

PROPOSITION 4.9. *Let A be an operator algebra, X, Y in ${}_A O_{\mathbb{C}}$. Then $E_{cb}xt^n(A - \mathbb{C}, \mathbb{C} - \mathbb{C}; Y, X)$ embeds as a subgroup of $H^n_{cb}(A; CB(Y, X))$.*

Proof. We have $E_{cb}xt^n(A - \mathbb{C}, \mathbb{C} - \mathbb{C}; Y, X) = H^n_{cb}(A; CB^{\ell}(Y, X))$, but since the matrix-norm structure on $CB^{\ell}(Y, X)$ dominates the matrix-norm structure on $CB(Y, X)$, we have that $Z^n_{cb}(A; CB^{\ell}(Y, X))$ is a subspace of $Z^n_{cb}(A; CB(Y, X))$ and $B^n_{cb}(A; CB^{\ell}(Y, X)) = B^n(A; CB(Y, X)) \cap Z^n(A; CB^{\ell}(Y, X))$ from which the result follows. ∎

Remark 4.10. Returning to the topic of Remark 3.4, and keeping the notation from there, we outline the parallel theory. First if one replaces the bar resolution at the beginning of this section by one using the operator space projective tensor norm and Hochschild cohomology defined by requiring all maps be jointly completely bounded, then one finds $E_{jcb}xt^n(A - B, C - D; Y, X) \cong H^n_{jcb}(A - B, C - D; Y, X)$ and in particular $H^n_{jcb}(A; X) \cong E_{jcb}xt^n(A - A, \mathbb{C} - A; A, X)$.

5. RELATIVE INJECTIVITY

In Section 4 we proved that completely bounded Hochschild cohomology is a relative Yoneda cohomology. This fact makes the role played by various relative notions of injective and projective modules central. In this section we focus on relative notions of injectivity.

We prove that, in an appropriate sense, every von Neumann algebra is relatively injective when viewed as an operator space bimodule over itself. This result has the immediate consequence that $H^n_{cb}(N; N) = 0$ for every von Neumann algebra N, which was first obtained in [CS3]. We also apply these ideas to give different proofs of results of [CS2] and [Pi] relating completely bounded projections to injectivity.

In a somewhat different direction we examine representations of the disk algebra on Hilbert space, regard the Hilbert column space as a left operator space module and relate injectivity of this module to variants of the classical Sz.-Nazy-Foias commutant lifting theory.

DEFINITION 5.1. An operator space I in ${}_A O_B$ will be called $A-B$-*injective* provided for every E, F in ${}_A O_B$ with $E \subseteq F$ a submodule and every φ in $CB_A(E, I)_B$ there exists ψ in $CB_A(F, I)_B$ which extends φ. Given subalgebras $C \subseteq A$, $D \subseteq B$, we will call $I (A-B, C-D)$-*injective* provided the same conclusions hold under the stronger hypothesis that the inclusion of E into F is $C-D$-split.

We call an A-bimodule I *relatively injective* if it is $(A-A, \mathbb{C}-\mathbb{C})$-injective.

In the above definition it clearly suffices to consider the case when E and F are completely contractive $A-B$-bimodules.

In each of the above definitions if, for E and F completely contractive, the stronger conclusion holds that ψ may be chosen with $\|\psi\|_{cb} = \|\varphi\|_{cb}$ then we add on the word *rigid*.

Note that the requirement that the inclusion of E into F is $C-D$-split, is equivalent to assuming that F is completely boundedly isomorphic to a module of the form $E \oplus_\gamma X$ for some X in ${}_A O_B$.

Remark 5.1. Let $C \subseteq C_1 \subseteq A$, $D \subseteq D_1 \subseteq B$ and let I be in ${}_A O_B$. Then it is readily seen that I $A-B$-injective implies that I is $(A-B, C-D)$-injective which in turn implies that I is $(A-B, C_1-D_1)$-injective.

Remark 5.2. Let $A, B \subseteq B(H)$ be C^*-subalgebras and regard $B(H)$ as in ${}_A O_B$. Then Wittstock's extension theorem [W] for $A-B$-bimodule maps says that $B(H)$ is a rigid $A-B$-injective module. (See [Su] for a simple proof of Wittstock's theorem.)

Remark 5.3. A C^*-algebra $A \subseteq B(H)$ is called *injective* (in the usual sense) if there exists a completely positive projection Φ: $B(H) \to A$. Such a map is automatically an A-bimodule map. Thus, by applying Remark 5.2 we see that A is injective if and only if it is a rigid $A-A$-injective module.

However, somewhat more is true as the following result shows. Recall that [BuPa] define a bounded map Q: $B(H) \to A$ to be a *quasi-expectation* provided that Q is an A-bimodule map which is also a projection, i.e., $Q(a) = a$ for all $a \in A$. They prove that if a von Neumann subalgebra of $B(H)$ has a quasi-expectation then A is injective. The following generalizes this to C^*-algebras.

THEOREM 5.4. *Let $1 \in A \subseteq B(H)$ be a C^*-algebra. The following are equivalent*:

(i) *A is an $A-A$-injective module,*

(ii) *there exists a quasi-expectation $Q: B(H) \to A$,*

(iii) *A is injective in the usual sense.*

Proof. By the same proof as [Pa1, Proposition 3.7] one sees that for any C^*-algebra A, every A-bimodule map ϕ with range A, satisfies $\|\phi\|_{cb} = \|\phi\|$. The equivalence of (i) and (ii) is a straightforward application of Remark 5.2, and clearly (iii) implies (ii). We prove that (ii) implies (iii).

To this end recall the existence and characterization of the injective envelope of a C^*-algebra due to Hamana [Ham]. If B denotes the injective envelope of A, then B is an injective C^*-algebra, $A \subseteq B$ and B is a *rigid* extension of A. This means [Ham] that, if $\Phi: B \to B$ is completely positive and $\Phi(a) = a$ for all a in A, then $\Phi(b) = b$ for all b in B.

The inclusion of A into $B(H)$ extends to a completely positive A-bimodule map $\phi: B \to B(H)$. Set $\gamma = Q \circ \phi$, so that $\gamma: B \to A$ is a completely bounded, A-bimodule projection onto A.

Note that $\gamma^*(b) = \gamma(b^*)^*$ is another completely bounded, A-bimodule projection onto A, and hence $\psi = (\gamma + \gamma^*)/2$ is a self-adjoint completely bounded A-bimodule projection onto A. By Wittstock's decomposition theorem for bimodule maps [W], there exists a completely positive A-bimodule map $\phi: B \to B$ with $\phi + \psi$ and $\phi - \psi$ completely positive.

Set $\phi(1) = p \geqslant \psi(1) = 1$. Since ϕ is an A-bimodule map, $ap = pa$ for all a in A. Let $\delta(b) = p^{-1/2}\phi(b)\, p^{-1/2}$, then δ is completely positive and $\delta(a) = a$ for every a in A. Hence, $\delta(b) = b$ and $\phi(b) = p^{1/2} b p^{1/2}$.

Similarly by considering for any real $t \geqslant 1$, $\delta_t(b) = (tp+1)^{-1/2} (t\phi(b) + \psi(b))(tp+1)^{-1/2}$, one finds $\delta_t(a) = a$ and δ_t is completely positive. Consequently

$$\psi(b) = (tp+1)^{1/2}\, b(tp+1)^{1/2} - tp^{1/2} b p^{1/2} \tag{$*$}$$

for any real $t \geqslant 1$. Since $p \geqslant 1$, the function $z \to (zp+1)^{1/2}$ has an analytic continuation into the region in the complex plane defined by $Re(z) > -1$. Thus for any fixed b in B the right-hand side of $(*)$ is analytic in this domain and constant for $z \geqslant +1$. Hence it is constant in the entire region. Evaluating at $z = 0$ we obtain $\psi(b) = b$.

Thus $\psi(b) = b$ for any b in B and ψ projects onto A, from which it follows that $B = A$. Hence A is injective as was to be shown. ∎

Later in this section we will prove that in contrast to the above, *every* von Neumann algebra is relatively injective as a bimodule over itself. We do not know which C^*-algebras enjoy this property. The situation for general operator algebras is far less clear. We do not know which subalgebras $A \subseteq B(H)$ have the property that $B(H)$ is $A-A$-injective or when there exists a completely bounded quasi-expectation. Clearly algebras similar either to C^*-algebras or to injective C^*-algebras, respectively, enjoy these properties.

The following result is immediate and summarizes the importance of the above definition.

PROPOSITION 5.5. *Let $C \subseteq A$, $D \subseteq B$ be unital operator algebras, and let I be in ${}_AO_B$. The following are equivalent:*

(i) *I is $(A-B, C-D)$-injective,*

(ii) *$E_{cb}xt^1(A-B, C-D; Y, I) = 0$ for every $Y \in {}_AO_B$,*

(iii) *$E_{cb}xt^n(A-B, C-D; Y, I) = 0$ for every $Y \in {}_AO_B$ and every n,*

(iv) *Every completely bounded $(A-B, C-D)$-derivation into I is inner.*

As an immediate consequence of these ideas we have the following proof of [CES].

PROPOSITION 5.6. *Let $A \subseteq B(H)$ be a C^*-algebra then $H^n_{cb}(A; B(H)) = 0$.*

Proof. $H^n_{cb}(A; B(H)) = E_{cb}xt^n(A-A, \mathbb{C}-A; A, B(H)) = 0$ because $B(H)$ is $A-A$-injective and hence $(A-A, \mathbb{C}-A)$-injective. ▮

One sees more clearly from the above exactly which properties of $B(H)$ are essential to the proof of the triviality. More generally if we regard H_c as a left A-module we have.

PROPOSITION 5.7. *Let $A \subseteq B(H)$ be a C^*-algebra then Hilbert column space, H_c is rigidly $A-\mathbb{C}$-injective and consequently,*

$$E_{cb}xt^n(A-\mathbb{C}, \mathbb{C}-\mathbb{C}; Y, H_c) = H^n_{cb}(A; CB^l(Y, H_c)) = 0$$

for $Y \in {}_AO_{\mathbb{C}}$.

Proof. Regard $H_c = B(H) \cdot E$ where E is a rank one projection. Then the inclusion $H_c \subseteq B(H)$ and projection given by right multiplication by E are $A-\mathbb{C}$-bimodule maps and so H_c is rigidly $A-\mathbb{C}$-injective and it follows by Proposition 5.5 that the first group is 0. The fact that the second group is 0 comes from applying Proposition 4.7. ▮

The next theorem is the principal theorem of this section.

THEOREM 5.8. *Every von Neumann algebra M is relatively injective as an $M-M$-bimodule.*

Proof. By Proposition 5.5 we must show that if Y is in ${}_M O_M$ and $\gamma: M \times Y \times M \to M$ is a completely bounded $(M-M, \mathbb{C}-\mathbb{C})$-derivation then there exists $\varphi \in CB(Y, M)$ such that $\gamma(a, y, b) = a\varphi(y)\, b - \varphi(ayb)$.

Let $M \subseteq B(H)$ be a weak*-closed subalgebra. By Remark 5.2. and Proposition 5.6, there exists $\varphi \in CB(Y, B(H))$ such that $\gamma(a, y, b) = a\varphi(y)\, b - \varphi(ayb)$, and we need to prove that we can replace φ by a map into M which implements γ.

By Proposition 2.1, we may assume that $Y \subseteq B(K)$ and that there is a *-homomorphism (not necessarily normal),

$$\theta: M \to B(K) \qquad \text{such that} \quad a \cdot y \cdot b = \theta(a)\, y\theta(b).$$

By extending φ to $B(K)$ and applying the generalized Stinespring representation of completely bounded maps [Pa1], we may write $\varphi(y) = V\pi(y)W$ where, $\pi: B(K) \to B(K_1)$ is a *-homomorphism and $W, V^* \in B(H, K_1)$.

Define $\rho: M \to B(K_1)$ by $\rho(m) = \pi(\theta(m))$ then we have $\gamma(a, y, b) = aV\pi(y)\, Wb - V\rho(a)\, \pi(y)\, \rho(b)\, W$ with $\rho(a)\, \pi(y)\, \rho(b) = \pi(ayb)$.

Suppose that we are given $a_1, ..., a_n, b_1, ..., b_m$ in M satisfying $\sum_{i=1}^n a_i^* a_i = 1 = \sum_{j=1}^m b_j b_j^*$ and we set $V_0 = \sum_{i=1}^n a_i^* V\rho(a_i)$, $W_0 = \sum_{j=1}^m \rho(b_1)\, Wb_j^*$. Then we will have that

$$\begin{aligned} V\pi(y)\, W - V_0\pi(y)\, W_0 &= \sum_{i,j} a_i^* [a_i V\pi(y)\, Wb_j - V\rho(a_i)\, \psi(y)\, \rho(b_j)\, W]\, b_j^* \\ &= \sum_{i,j} a_i^* \gamma(a_i, y, b_j)\, b_j^* \end{aligned}$$

which is clearly in M. Hence setting $\psi(y) = V\pi(y)\, W - V_0\pi(y)\, W_0$ defines a completely bounded map from Y into M with $\|\psi\|_{cb} \leqslant 2\, \|V\|\, \|W\| = 2\, \|\varphi\|_{cb}$.

Furthermore, if we replaced V_0 and W_0 by a weak*-limit of operators of the above form, then the map ψ would still have range contained in M.

Note that for ψ to implement γ we would need that

$$0 = aV_0\pi(y)\, W_0 b - V_0\rho(a)\, \pi(y)\, \rho(b)\, W_0$$

which would follow if $aV_0 = V_0\rho(a)$ and $\rho(b)\, W_0 = W_0 b$ for all a, b in M.

We will prove, via some modifications of the arguments in [CS3] that there do exist operators V_1 and W_1 which are weak*-limits of operators of the form V_0 and W_0 and which satisfy $aV_1 = V_1\rho(a)$, $W_1 b = \rho(b)\, W_1$, and this fact will complete the proof of the theorem.

First to construct V_1 one looks at the set

$$\mathscr{S} = \{(a_1, a_2, ...) \in \ell^\infty(M) \colon \sum_{i=1}^\infty a_i^* a_i = 1\}$$

where the series converges strongly. Each $a=(a_1, a_2, ...)$ in $\mathscr{S}$ defines a map $\alpha_a\colon B(K_1, H)\to B(K_1, H)$ via $\alpha_a(T)=\sum a_i^* T\rho(a_i)$ and $\mathscr{S}$ with this action defines a convex semigroup of completely contractive mappings on $B(K_1, H)$. One easily checks that T is fixed by this semigroup if and only if $aT=T\rho(a)$ for all $a\in M$.

If we let $\mathscr{C}$ denote the weak*-closure of the orbit of V under the action of this semigroup, then $\mathscr{C}$ is bounded by $\|V\|$, and hence compact and convex in the weak*-topology. Thus, there exist minimal non-empty weak*-compact convex $\mathscr{S}$-invariant subsets of $\mathscr{C}$. Let $\mathscr{C}_0$ denote such a set and we shall prove that for any $R\in\mathscr{C}_0$, $aR=R\rho(a)$ and hence any element of $\mathscr{C}_0$ will do for V_1. (Since $\mathscr{C}_0$ is minimal it must, consequently, be the case that $\mathscr{C}_0$ consists of a single operator.)

First some elementary reductions. Note that the closed span $\{\pi(y)\, Wh\colon y\in Y,\ h\in H\}$ is invariant and hence reducing for $\rho(M)$. Thus, we may cut down by the projection onto this space and consequently we may assume that this span is dense in K_1.

Following the proof in [CS3] we first establish that if e is any projection in M, and $C(e)$ denotes its central support, then for any $R\in\mathscr{C}_0$,

$$\|eR\|=\|eR\rho(e)\|=\|C(e)\, R\| \tag{1}$$

To see this first note that for any g in the commutant of M and h in the commutant of $\rho(M)$ the function on $\mathscr{C}_0$, $R\to\|gRh\|$ must be constant. For suppose not, then there would exist some number r such that the set $\mathscr{D}=\{R\in\mathscr{C}_0\colon \|gRh\|\leqslant r\}$ is non-empty and not all of $\mathscr{C}_0$. Clearly $\mathscr{D}$ is weak*-closed and convex. It remains to show that $\mathscr{D}$ is S-invariant which would contradict the minimality of $\mathscr{C}_0$. But if $R\in\mathscr{C}_0$, $\|gRh\|<r$ and $a=(a_1, a_2, ...)$ is in $\mathscr{S}$, then

$$\left\|g\left(\sum a_i^* R\rho(a_i)\right)h\right\|=\left\|\sum a_i^*(gRh)\,\rho(a_i)\right\|\leqslant\|gRh\|\leqslant r$$

and hence $\mathscr{D}$ is $\mathscr{S}$-invariant.

Now we may choose a family of partial isometries $\{v_j\}$ in M with $v_j^* v_j$ pairwise orthogonal, $v_j v_j^*\leqslant e$ and $\sum v_j^* v_j=C(e)$. Setting $a_1=1-C(e)$, and $a_n=v_{n-1}, n\geqslant 2$ defines an element a of $\mathscr{S}$ and hence for R in $\mathscr{C}_0$ with $g=C(e)$, $h=1$, we have

$$\begin{aligned}\|C(e)R\|&=\|C(e)\,\alpha_a(R)\|\\&=\left\|C(e)(1-C(e))\,R\rho(1-C(e))+\sum C(e)\,v_j^* R\rho(v_j)\right\|\\&=\left\|\sum v_j^* R\rho(v_j)\right\|=\sup\|v_j^* R\rho(v_j)\|\end{aligned}$$

since the supports and ranges of the operators $v_j^* R\rho(v_j)$ are orthogonal. Now $ev_j = v_j$ and $\|v_j\| = 1$ implies that $\|v_j^* R\rho(v_j)\| = \|v_j^* eR\rho(e) \cdot \rho(v_j)\| \leqslant \|eR\rho(e)\|$. Hence, $\|C(e)R\| \leqslant \|eR\rho(e)\| \leqslant \|eR\| \leqslant \|C(e)R\|$ from which (1) follows.

Now assume that for some projection e in M, y in Y and R in $\mathscr{C}_0$,

$$b = eR\rho(1-e)\,\pi(y)\,W \neq 0. \tag{2}$$

Note that if $R = \sum a_i^* V\rho(a_i)$ then

$$\begin{aligned} b &= \sum ea_i^* V\rho(a_i)\,\rho(1-e)\,\pi(y)\,W = \sum ea_i^*\,\varphi(a_i(1-e)\,y) \\ &= \sum ea_i^*\,(a_i\varphi((1-e)\,y) - \gamma(a_i, (1-e)\,y, 1)) \\ &= e\varphi((1-e)\,y) - m_1 = \varphi(e(1-e)\,y) + \gamma(e, (1-e)\,y, 1) \\ &\quad - m_1 = \gamma(e, (1-e)\,y, 1) - m_1 \qquad \text{for some } m_1 \text{ in } M. \end{aligned}$$

Thus, in this case we would have b in M. Since every R in $\mathscr{C}_0$ is a weak*-limit of R's of the above form, we have that the element b defined by (2) is in M.

Since $bb^* \neq 0$, we may choose f in M to be the spectral projection of bb^* for the interval $[\,\|b\|^2/2, \|b\|^2]$. Note that since $eb = b$, we have that $f \leqslant e$. By (1), $\|R^*f\| = \|\rho(f)\,R^*f\|$ and hence given $\varepsilon > 0$ we may choose a vector h, $\|h\| = 1$ and $fh = h$, satisfying

$$\begin{aligned} (1-\varepsilon)\,\|R^*f\|^2 &< \|\rho(f)\,R^*fh\|^2 \leqslant \|\rho(e)\,R^*fh\|^2 \\ &= \|R^*fh\|^2 - \|(1-\rho(e))\,R^*fh\|^2 \\ &\leqslant \|R^*fh\|^2 - \|W^*\pi(y)^*\|^{-2}\,\|W^*\pi(y)^*(1-\rho(e))\,R^*fh\|^2 \\ &\leqslant \|R^*fh\|^2 - \|W^*\pi(y)^*\|^{-2}\,\|b^*fh\|^2 \\ &\leqslant \|R^*fh\|^2 - \|W^*\pi(y)^*\|^{-2}\,\|b\|^2/2. \end{aligned}$$

Set $\varepsilon = \|b\|^2\,(2\,\|\pi(y)\,W\|\,\|R^*f\|)^2$, then this last term becomes $\|R^*f\|^{-2} - 2\varepsilon\,\|R^*f\|^2$. We have $(1-\varepsilon)\,\|R^*f\|^2 \leqslant (1-2\varepsilon)\,\|R^*f\|^2$. Since $\varepsilon > 0$ we must have that $\|R^*f\| = 0$. But then $0 = fR = feR$ and so $fb = 0$. This contradicts the choice of f and so $b = 0$.

Thus, $eR\rho(1-e)\,\pi(y)\,W = 0$ for every $y \in Y$, but by the density of the span in K_1 we must have that $eR\rho(1-e) = 0$ for every projection e in M and R in $\mathscr{C}_0$. Since $eR = eR\rho(e)$ for every projection, we have that $R\rho(e) = (e + (1-e))\,R\rho(e) = eR\rho(e) = eR$ for all projections. Now since every unitary in M is a norm limit of linear combinations of spectral projections, we have that $uR = R\rho(u)$ for every unitary $u \in M$ and hence for every element of M. Thus, $aR = R\rho(a)$ for every a in M as was to be shown.

One constructs W_1 in a similar manner. We discuss the key steps. First let $\mathcal{S}=\{(b_1, ...): \sum b_i b_i^* = 1\}$ with the series converging strongly, and let this semigroup act on $B(H, K_1)$ and again choose a minimal nonempty weak*-compact convex set, $\mathcal{C}_0$ from the orbit of W. For this set one establishes:

$$\|Te\| = \|\rho(e)\, Te\| = \|TC(e)\|. \tag{1'}$$

Next one assumes for some y in Y and T in $\mathcal{C}_0$,

$$a = V\pi(y)\, \rho(1-e)\, Te \neq 0. \tag{2'}$$

To see that a is in M one checks first the case when $T = \sum \rho(b_i)\, Wb_i^*$. As above one concludes that $a=0$ and by the density of $\{\pi(y)^* V^* h: y \in Y, h \in H\}$ in K_1, which can be assumed by again cutting down by an invariant projection, one finds $\rho(1-e)\, Te = 0$ for all $T \in \mathcal{C}_0$ and projection e. Again this implies that $\rho(a)T = Ta$, which completes the proof of the theorem. ∎

Theorem 5.9. *Every von Neumann algebra M is $(M-M, \mathbb{C}-M)$-injective, $(M-M, M-\mathbb{C})$-injective, $(M-\mathbb{C}, \mathbb{C}-\mathbb{C})$-injective and $(\mathbb{C}-M, \mathbb{C}-\mathbb{C})$-injective, when regarded as either an M-bimodule, left M-module or right M-module, respectively.*

Proof. The first two claims follow from Theorem 5.8 and Remark 5.1. The second two claims follow by imitating the proof of Theorem 5.8. ∎

We now have a new proof of [CS3], see also [SS, Theorem 4.3.1].

Corollary 5.10. *Let M be a von Neumann algebra then $H^n_{cb}(M; M) = 0$.*

Proof. $H^n_{cb}(M; M) = E_{cb}xt^n(M-M, \mathbb{C}-M; M, M) = 0$ since the second factor M is $(M-M, \mathbb{C}-M)$ injective. ∎

We can also give a somewhat different proof of [CS3] and [Pi].

Corollary 5.11. *Let $M \subseteq B(H)$ be a von Neumann subalgebra. If there exists a completely bounded projection $P: B(H) \to M$, then M is injective.*

Proof. The existence of P implies that the inclusion of M into $B(H)$ is $\mathbb{C}-\mathbb{C}$-split. Since M is relatively injective, the identity map from M to M extends to a completely bounded M-bimodule map $Q: B(H) \to M$. Thus, Q is a completely bounded quasi-expectation and applying [BuPa, Theorem 2] or Theorem 5.4 we have that M is injective. ∎

The above proof also applies to relatively injective C^*-algebras. We close this section with some examples motivated more by operator theory and the Sz.-Nagy-Foias commutant lifting theorem, by looking at operator space modules over the disk algebra, $A(\mathbb{D})$.

THEOREM 5.12. *Let W be a coisometry and V be an isometry on a Hilbert space H and regard $B(H)$ as an $A(\mathbb{D})$-bimodule with left action given by W and right action given by V. Then $B(H)$ is $A(\mathbb{D})-A(\mathbb{D})$-injective.*

Proof. We first consider the case of $B(H)$ regarded as a left module and prove that it is $A(\mathbb{D})-\mathbb{C}$-injective.

It suffices to assume that $X \subseteq Y$ are completely contractive left $A(\mathbb{D})$-modules and that $\psi: X \to B(H)$ is completely bounded with $\psi(z \cdot x) = W\psi(x)$ for all $x \in X$ where z denotes the coordinate function. Applying the Stinespring representation to ψ, yields a completely contractive representation π of Y as operators on a Hilbert space K, $\pi(Y) \subseteq B(K)$, a contraction operator $T \in B(K)$ with $T \cdot \pi(y) = \pi(z \cdot y)$ and operators $R, S^* \in B(K, H)$ such that $\psi(x) = R\pi(x)S$, for all $x \in X$, $y \in Y$.

Let K_1 denote the closed subspace of K spanned by vectors of the form $\pi(x)\, Sh$, for $x \in X, h \in H$. Then K_1 is T-invariant and since $RT\pi(x)\, S = \psi(z \cdot x) = WR\pi(x)\, \mathbb{S}$ we have that $RTk = WRk$ for k in K_1. Let R_1 denote the restriction of R to K_1, and T_1 denote the compression of T to K_1, so that $R_1 T_1 = WR_1$. Assume for the moment that we could extend R_1 to $B: K \to H$ with $BT = WB$. Then setting $\psi'(y) = B\pi(y)\, S$ yields the completely bounded $A(\mathbb{D})$-module extension to Y.

Taking adjoints we have that W^* is an isometry on H, T^* makes K an $A(\mathbb{D})$-module, for which K_1 is a quotient $A(\mathbb{D})$ module with action arising from T_1^* and $T_1^* R_1^* = R_1^* W^*$.

By a variant of the Sz.-Nagy-Foias commutant lifting [DF], there exists an operator $B^*: H \to K$ with $T^*B^* = B^*W^*$, see also [DP, Theorem 4.11]. This completes the proof for the left action.

The case for a right action by an isometry follows by taking adjoints, and the bimodule case by applying the above argument to both actions. ∎

COROLLARY 5.13. *Let V be an isometry on K and W a coisometry on H and regard $B(K, H)$ as a completely contractive $A(\mathbb{D})$-bimodule. Then $B(K, H)$ is $A(\mathbb{D})-A(\mathbb{D})$-injective and consequently*

$$H^n_{cb}(A(\mathbb{D}); B(K, H)) = 0 \qquad \textit{for all } n.$$

Recall the definitions of Hilbert column H_c and Hilbert row space H_r as subspaces of $B(H)$. If $A \subseteq B(H)$ is a subalgebra then H_c is a left A-submodule of $B(H)$ while H_r is a right A-submodule.

COROLLARY 5.14. *Let V be a coisometry (respectively, isometry) on a Hilbert space H and let H_c (respectively, H_r) denote the corresponding left (respectively, right) $A(\mathbb{D})$-module. Then H_c (respectively, H_r) is $A(\mathbb{D})-\mathbb{C}$-injective (respectively, $\mathbb{C}-A(\mathbb{D})$-injective).*

Proof. The projection from $B(H)$ onto H_c is a left $A(\mathbb{D})$-module map. Consequently, $A(\mathbb{D})-\mathbb{C}$-injectivity of $B(H)$ implies it for H_c. The H_r case follows similarly.

The above result allows us to give a homological proof of a result of Foias–Williams [FW] (see also [CCFW]).

COROLLARY 5.15. *Let $R=\left(\begin{smallmatrix} V & X \\ 0 & W\end{smallmatrix}\right)$ be an operator on the direct sum of two Hilbert spaces, $H\oplus K$, with V and W similar to contractions and either V a coisometry or W an isometry. Then R is similar to a contraction if and only if $X=VS-SW$ for some $S\in B(K,H)$.*

Proof. First consider the case where V is a coisometry. If X has the above form

$$\begin{pmatrix} 1 & +S \\ 0 & 1\end{pmatrix}\begin{pmatrix} V & X \\ 0 & W\end{pmatrix}\begin{pmatrix} 1 & -S \\ 0 & 1\end{pmatrix}=\begin{pmatrix} V & 0 \\ 0 & W\end{pmatrix},$$

and so R is similar to the direct sum of V and W, which is in turn similar to a contraction.

Conversely, if R is similar to a contraction, then $z\to R$ defines a completely bounded representation of $A(\mathbb{D})$ onto $B(H\oplus K)$. This makes $H_c\oplus K_c$ a completely bounded left $A(\mathbb{D})$-module, with H_c an $A(\mathbb{D})$-submodule. By 5.13 the sequence of left operator space $A(\mathbb{D})$-modules,

$$0\to H_c\to H_c\oplus K_c\to K_c\to 0$$

must split. Hence there exists $T\in CB(H_c\oplus K_c, H_c)=B(H\oplus K, H)$ which fixes H and is a left $A(\mathbb{D})$-module map. Thus, $T(h\oplus k)=h+Sk$ for some $S\in B(K,H)$ and $VT(h\oplus k)=TR(h\oplus k)$, which implies $X=VS-SW$.

The case when W is an isometry follows by either considering R^* or regarding $H_r\oplus K_r$ as a right $A(\mathbb{D})$-module with K_r a submodule under right multiplication by R. ∎

6. RELATIVE PROJECTIVITY

In this section we focus on projectivity and relative projectivity. We prove that the only C^*-algebras that have the property that $H^1_{cb}(A;X)=0$ for all X in ${}_A O_A$ are the finite direct sums of matrix algebras. This is analogous to theorems of Helemskii [He] and Selivanov [Se] in the Banach algebra setting.

DEFINITION 6.1. An operator space P in ${}_A O_B$ will be called $A-B$-*projective* provided for every E, F in ${}_A O_B$ with $E \subseteq F$ a submodule and every φ in $CB_A(P, F/E)_B$ there exists $\psi \in CB_A(P, F)_B$ such that $q \circ \psi = \varphi$ where $q: F \to F/E$ denotes the quotient map. Given $C \subseteq A$, $D \subseteq B$ we will call P $(A-B, C-D)$-*projective* provided the same conclusion holds under the stronger hypothesis that the quotient map is $C-D$-split.

Remark 6.2. If $C \subseteq C_1 \subseteqq A$, $D \subseteq D_1 \subseteq B$, then we have that $A-B$-projective implies $(A-B, C-D)$-projective which in turn implies $(A-B, C_1-D_1)$-projective.

Remark 6.3. It follows readily from the definition that every term in the $(A-B, C-D)$-bar resolution is $(A-B, C-D)$-projective. Thus, this resolution is a projective resolution in a sense appropriate to our theory.

The following is elementary.

PROPOSITION 6.4. *Let $1 \in C \subseteq A$, $1 \in D \subseteq B$ be operator algebras and let P be in ${}_A O_B$. Then the following are equivalent*:

(i) *P is $(A-B, C-D)$-projective*;

(ii) *$E_{cb}xt^1(A-B, C-D; P, Y) = 0$ for all Y in ${}_A O_B$*;

(iii) *$E_{cb}xt^n(A-B, C-D; P, Y) = 0$ for all Y in ${}_A O_B$ and all n*;

(iv) *for every Y in ${}_A O_B$, every completely bounded $(A-B, C-D)$-derivation $\gamma: A \times P \times B \to Y$ is inner*;

(v) *the map $\pi_0: A \otimes_{h, C} P \otimes_{h, D} B \to P$ is (A, B)-split*; *and*

(vi) *$E_{cb}xt^1(A-B, C-D; P, \ker \pi_0) = 0$*

DEFINITION 6.5. Let $1 \in C \subseteq A$ be operator algebras. An element u in $A \otimes_{hC} A$ is called a *C-relative diagonal* provided that $a \cdot u = u \cdot a$ for all a in A and $\pi_0(u) = 1_A$ where $\pi_0: A \otimes_{hC} A \to A$ is the product map. When $C = \mathbb{C}$ we will simply call u a *diagonal*.

THEOREM 6.6. *Let $1 \in C \subseteq A$ be operator algebras. Then the following are equivalent*:

(i) *A is an $(A-A, C-A)$-projective A-bimodule*;

(ii) *$H^1_{cb}(A, C; Y) = 0$ for every Y in ${}_A O_A$*;

(iii) *$H^n_{cb}(A, C; Y) = 0$ for every Y in ${}_A O_A$*;

(iv) *For every* Y *in* ${}_AO_A$ *and for every completely bounded* $\delta\colon A \to Y$ *satisfying* $\delta(ab) = a\delta(b) + \delta(a)b$, $\delta(c) = 0$, a, b *in* A, c *in* C, *there exists a* y *in* Y *with* $\delta(a) = ay - ya$; *and*

(v) A *has a* C*-relative diagonal.*

Proof. Clearly (i), (ii), and (iii) are just 6.4(i), 6.4(ii), 6.4(iii) when we take $P = A = B$. Similarly, (iv) is just 6.4(iv) when we observe that any $\gamma\colon A \times A \times A \to Y$ which is a completely bounded $(A - A,\ C - A)$-derivation gives rise to $\delta\colon A \to Y$ by setting $\delta(a) = \gamma(a, 1, 1)$. Conversely, given δ one obtains γ with the desired properties by setting $\gamma(a_1, a_2, a_3) = \delta(a_1)\, a_2 a_3$. Now one checks that γ is inner if and only if there exists y in Y with $\delta(a) = a \cdot y - y \cdot a$. In fact, if $\gamma(a_1, a_2, a_3) = a_1 \varphi(a_2)\, a_3 - \varphi(a_1 a_2 a_3)$ then $y = \varphi(1)$.

Finally, one notes that the existence of a C-relative diagonal is equivalent to 6.4(v) by setting $u = \varphi(1)$ where $\varphi\colon A \to A \otimes_{h,\,C} A \otimes_{h,\,A} A (= A \otimes_{h,\,C} A)$ is the $A - A$-splitting of π_0. ∎

To see the importance of C-relative diagonals, suppose that our algebra A contains a (norm) compact group of invertibles G which normalize C, i.e., $g^{-1}Cg \subseteq C$ and such that the algebra generated by C and G is dense in A. This is the case, for example, when C is a C^*-algebra and $A = C \times_\alpha G$ for some finite group G. If for c in C, $g^{-1}cg = d$ in C then $cg \otimes_C g^{-1} = gd \otimes_C g^{-1} = g \otimes_C dg^{-1} = g \otimes_C g^{-1}c$. Hence the element $u = \int_G g \otimes_{h,\,C} g^{-1}\, dg$ is easily seen to be a C-relative diagonal in $A \otimes_{h,\,C} A$ and consequently $H^n(A, C; Y) = 0$ for any Y in ${}_AO_A$.

Remark 6.7. It is known, see for example [He] that the vanishing of the bounded Hochschild cohomology groups for a Banach algebra A is equivalent to the existence of a diagonal u in the projective tensor product of A with itself. If A is an operator algebra, then since the projective tensor norm dominates the Haagerup tensor norm this element will define a diagonal in our sense as well. However, since the Haagerup norm is smaller, one might expect more algebras to have a diagonal in our sense. We prove in Theorem 6.13 that the only C^*-algebras which possess a diagonal in the Haagerup norm are the finite dimensional ones. Since these possess a diagonal in the algebraic tensor product, it then follows that the only C^*-algebras which possess a diagonal in the projective tensor product are again the finite dimensional ones, giving a new proof of a result first obtained by Selivanov [Se].

In the remainder of this section we examine the ramifications of A being $(A - A,\ \mathbb{C} - A)$-projective, i.e., of having a diagonal in $A \otimes_h A$.

PROPOSITION 6.8. *Let A be an operator algebra, $\rho: A \to B(H)$ a completely bounded homorphism and let $B = \rho(A)'$ denote the commutant. If A has a diagonal, then there is a completely bounded projection, $\Phi: B(H) \to B$ which is a B-bimodule map.*

Proof. Let $w = \sum a_i \otimes b_i$ be in $A \otimes_h A$ and define $\Phi_w: B(H) \to B(H)$ via $\Phi_w(T) = \sum \rho(a_i)\, T\rho(b_i)$ so that $\|\Phi_w\|_{cb} \leqslant \|\rho\|_{cb}^2 \|w\|$. Clearly, Φ_w is a B-bimodule map and $\rho(a)\, \Phi_w(T) = \Phi_{a \cdot w}(T)$, $\Phi_w(T)\, \rho(a) = \Phi_{w \cdot a}(T)$.

Thus, when u is a diagonal we have, $\rho(a)\, \Phi_u(T) = \Phi_u(T)\, \rho(a)$ and so the range of Φ_u is contained in B. Also, for T in B we have $\Phi_u(T) = T$. Thus, Φ_u is the desired completely bounded module projection. ∎

PROPOSITION 6.9. *Let A be an operator algebra, let $u = \sum a_i \otimes b_i$ be a diagonal and let $\rho: A \to B(H)$ be a unital completely bounded homomorphism. Then $\sum \rho(a_i) \otimes \rho(b_i)$ is a diagonal for $\rho(A)''$.*

Proof. Let $w = \sum \rho(a_i) \otimes \rho(b_i)$, and let $\Phi_w: B(H) \to \rho(A)'$ be defined by $\Phi_w(T) = \sum \rho(a_i)\, T\rho(b_i)$. For any R in $\rho(A)''$ we have that $\Phi_{R \cdot w} = \Phi_{w \cdot R}$. Since the inclusion of $\rho(A)'' \otimes_h \rho(A)''$ into $B(H) \otimes_h B(H)$ is a complete isometry [PS] and the map which sends v in $B(H) \otimes_h B(H)$ to Φ_v in $CB(B(H), B(H))$ is one-to-one we have that $R \cdot w = w \cdot R$ as elements of $\rho(A)'' \otimes_h \rho(A)''$. Clearly, $\sum \rho(a_i)\, \rho(b_i) = 1$ and so w is a diagonal in $\rho(A)''$. ∎

COROLLARY 6.10. *Let A be a unital operator algebra, let $\rho: A \to B(H)$ be a unital completely bounded homomorphism and let $B = \rho(A)''$. If $H^1_{cb}(A; X) = 0$ for all X in ${}_A\mathcal{O}_A$, then $H^n_{cb}(B; Y) = 0$ for all Y in ${}_B\mathcal{O}_B$ and all n.*

PROPOSITION 6.11. *Let A be a unital operator algebra, I a closed 2-sided ideal in A. If A has a diagonal in $A \otimes_h A$ then A/I has a diagonal in $(A/I) \otimes_h (A/I)$. Consequently, if $H^1_{cb}(A; X) = 0$ for all X in ${}_A\mathcal{O}_A$, then $H^n_{cb}(B; Y) = 0$ for all Y in ${}_B\mathcal{O}_B$ and all n, where $B = A/I$.*

Proof. If $u = \sum a_i \otimes b_i$ is a diagonal in $A \otimes_h A$ then $\sum \pi(a_i) \otimes \pi(b_i)$ is a diagonal in $B \otimes_h B$. ∎

PROPOSITION 6.12. *Let A be a unital commutative operator algebra, for which the Gelfand transform is one-to-one. If A has a diagonal in $A \otimes_h A$ then A is completely boundedly isomorphic to ℓ^∞_n for some n.*

Proof. Let X denote the maximal ideal space, $\pi: A \to C(X)$ the Gelfand transform and let $u = \sum a_i \otimes b_i$ be a diagonal in $A \otimes_h A$. Set $f_i = \pi(a_i)$, $g_i = \pi(b_i)$, then since π is completely contractive the series $h(x, y) = \sum_i f_i(x)\, g_i(y)$ converges uniformly to a continuous function on $X \times X$.

Since $\pi(A)$ separates points on X, one sees readily that h is necessarily the characteristic function of the diagonal D. Hence D is both open and closed in $X \times X$, which implies that X is discrete and consequently finite.

Hence, $C(X) = \ell_n^\infty$ for some n and the Gelfand transform is an algebraic isomorphism. Thus, A is finite dimensional but by [Pa3] any bounded map between finite dimensional operator spaces is completely bounded. ∎

We now turn our attention to the case of C^*-algebras.

THEOREM 6.13. *Let A be a unital C^*-algebra. If A has a diagonal in $A \otimes_h A$, then A is *-isomorphic to a finite direct sum of matrix algebras.*

Proof. Let $\rho: A \to B(H)$ be a unital *-homomorphism. By 6.8, there is a completely bounded projection from $B(H)$ onto the von Neumann algebra $\rho(A)'$. By 5.4 this implies that $\rho(A)'$ is injective. The fact that $\rho(A)'$ is injective for every ρ, implies that A is nuclear [CE1, CE2].

However, if $B = \rho(A)''$ then B is an injective von Neumann algebra, since $\rho(A)'$ is injective. But by 6.9, $B \otimes_h B$ also possesses a diagonal and so by the above B is nuclear.

Invoking either [SW, Corollary 6.8] or [W], we have that the only nuclear injective von Neumann algebras are finite direct sums of algebras of the form $C(X) \otimes M_n$, with X Stonean.

By 6.11 we have that each of the summands would posses a diagonal.

Now let $u = \sum f_i \otimes g_i$ be a diagonal in $(C(X) \otimes M_n) \otimes_h (C(X) \otimes M_n)$ and consider the matrix valued function $h(x, y) = \sum f_i(x)\, g_i(y)$. As in the proof of 6.12 one sees that this converges uniformly to define a continuous matrix-valued function in $C(X \times X) \otimes M_n$, which in fact is the characteristic function of the diagonal in $X \times X$ tensored with the identity matrix. Thus, as before X is a finite set from which the result follows. ∎

COROLLARY 6.14. *Let A be a C^*-algebra. Then $H^1_{cb}(A; X) = 0$ for all X in ${}_A O_A$ if and only if A is a finite direct sum of matrix algebras.*

Remark 6.15. R. Smith has recently given a proof, using only properties of the Haagerup tensor product, that if an operator algebra possesses a diagonal in $A \otimes_h A$ then it is isomorphic to a finite direct sum of matrix algebras.

Remark 6.16. In an analogous fashion one has $H^1_{jcb}(A; X) = 0$ for every operator space X which is a jointly completely bounded A-bimodule if and only if A possesses a diagonal in $A \widehat{\otimes} A$-the operator space projective tensor product. Since this tensor norm lies between the Haagerup and Banach space projective tensor norms, we see that the only C^*-algebras possessing a diagonal here are again finite direct sums of matrix algebras.

7. APPROXIMATE DIAGONALS AND AMENABILITY

In this section we turn our attention to an appropriate notion of amenability. Recall that if a Banach space Y is a bounded $B-A$-bimodule then the dual of Y, Y^* is an $A-B$-bimodule under the action $(a \cdot f \cdot b)(x) = f(bxa)$. If Y is an operator space and A, B are operator algebras then generally assuming that Y is in ${}_B O_A$ does not guarantee that Y^* is in ${}_A O_B$. For this reason we define an operator space X to be a *completely bounded dual $A-B$-bimodule* if and only if there exists an operator space Y which is a bounded $B-A$-bimodule such that $X = Y^*$ and X equipped with the induced $A-B$-bimodule action is in ${}_A O_B$.

DEFINITION 7.1. Let $1 \in C \subseteq A$, $1 \in D \subseteq B$ be operator algebras. An operator space E in ${}_A O_B$ is called $(A-B, C-D)$-*amenable* provided $E_{cb}xt^1(A-B, C-D; E, X) = 0$ for every completely bounded dual $A-B$-bimodule X.

Note that every $(A-B, C-D)$-projective module is $(A-B, C-D)$-amenable.

The following result is more subtle than it may at first appear.

PROPOSITION 7.2. *Let A, B be unital operator algebras and let X be in ${}_A O_B$. Then X^{**} is a completely bounded dual $A-B$-module, and $A^{**}-B^{**}$-bimodule.*

Proof. Up to a completely bounded isomorphism of X we may assume that A, B are subalgebras of $B(H)$ and that X is a subspace of $B(H)$ with $AXB \subseteq X$. Consider the concrete operator algebra $C = \{\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) : a \in A, b \in B, x \in X\}$ contained in $B(H \oplus H)$. By [Bl1], C^{**} with the usual Arens multiplication is an operator algebra, containing C as a subalgebra. One easily checks that as an algebra, $C^{**} = \{\left(\begin{smallmatrix} a^{**} & x^{**} \\ 0 & b^{**} \end{smallmatrix}\right) : a^{**} \in A^{**}, b^{**} \in B^{**}, x^{**} \in X^{**}\}$ completely isomorphically with the containment of C in C^{**} the obvious containment. Consequently, X^{**} is a completely contractive $A^{**}-B^{**}$-bimodule. ∎

DEFINITION 7.3. Let A be a not necessarily unital algebra. A net $\{\mu_\alpha\}$ in $A \otimes_h A$ is called an *approximate diagonal* provided $\{\|\mu_\alpha\|\}$ is bounded, $\|a \cdot \mu_\alpha - \mu_\alpha \cdot a\| \to 0$ and $\|\pi(\mu_\alpha)\, a - a\| \to 0$ for every a in A, where $\pi: A \otimes_h A \to A$ denotes the product map. An element $v \in (A \otimes_h A)^{**}$ is called a *virtual diagonal* provided $a \cdot v = v \cdot a$ and $\pi^{**}(v) a = a$ for all $a \in A$.

Note that if we require our algebra to be unital we may and do assume that $\pi(\mu_\alpha) = 1$ for all α. Also, the existence of an approximate diagonal guarantees that A has a bounded 2-sided approximate identity.

For an arbitrary operator algebra with a bounded, 2-sided approximate identity, A we set $A_+ = A$ when A is unital and otherwise we let A_+ denote the algebra obtained from A by adjoining an identity.

THEOREM 7.4. *Let A be an operator algebra with a contractive 2-sided approximate identity. Then the following are equivalent*:

(i) *A is $(A_+ - A_+, \mathbb{C} - A_+)$-amenable*;

(ii) *A has an approximate diagonal in $A \otimes_h A$; and*

(iii) *A has a virtual diagonal in $(A \otimes_h A)^{**}$.*

Proof. The equivalence of (ii) and (iii) follows as in [Jo].

Now assume that A has an approximate diagonal $\{\mu_\alpha\}$ in $A \otimes_h A$. We must prove that every $\gamma: A_+ \times A \times A_+ \to X$ is inner whenever X is a completely bounded dual A_+-bimodule and γ is $\mathbb{C} - A_+$-split. Note $\gamma(a, b, c) = \gamma(a, b, 1)\, c$ for all $c \in A_+$ and $A_+ \otimes_{h, \mathbb{C}} A \otimes_{h, A_+} A_+ = A_+ \otimes_h A$.

Note also that we may extend γ to $A_+ \times A_+ \times A_+$ by letting e_α be a contractive 2-sided approximate identity and defining

$$\gamma(a, b, c) = \lim_\alpha \gamma(a, e_\alpha b, c).$$

Consequently,

$$\gamma(a, b, d) = \lim_\alpha \gamma(a, e_\alpha b, c) = \lim_\alpha \gamma(a, e_\alpha, bc) = \gamma(a, 1, 1)\, bc.$$

Let $\Gamma: A_+ \otimes_h A \to X$ be the linearization of γ, and let $\mu_\alpha = \sum b_n^\alpha \otimes c_n^\alpha$. Since $\|a \cdot \mu_\alpha - \mu_\alpha a\| \to 0$ we have that $\|\Gamma(a \cdot \mu_\alpha) - \Gamma(\mu_\alpha \cdot a)\| \to 0$ but

$$\Gamma(a \cdot \mu_\alpha) - \Gamma(\mu_\alpha \cdot a) = \sum \gamma(ab_n^\alpha, c_n^\alpha, 1) - \gamma(b_n^\alpha, c_n^\alpha a, 1)$$

$$= \sum (a\gamma(b_n^\alpha, c_n^\alpha, 1) + \gamma(a, 1, 1)\, b_n^\alpha c_n^\alpha - \gamma(b_n^\alpha, c_n^\alpha, 1)\, a)$$

using the derivation property and $\mathbb{C} - A$-modularity.

Let $f_\alpha = \sum \gamma(b_n^\alpha, c_n^\alpha, 1)$ so that $\Gamma(a \cdot \mu_\alpha) - \Gamma(\mu_\alpha \cdot a) = a \cdot f_\alpha - f_\alpha \cdot a + \gamma(a, 1, 1)\, \pi(\mu_\alpha)$. Let g be any weak*-limit point of $\{f_\alpha\}$, then using the fact that $x \cdot \pi(\mu_\alpha)$ weak*-converges to x we have

$$a \cdot g - g \cdot a + \gamma(a, 1, 1) = 0 \qquad \text{for all } a \text{ in } A.$$

Thus, defining $\varphi: A \to X$ by $\varphi(a) = g \cdot a$ we have that $\gamma(a_1, a, a_2) = \gamma(a_1, 1, 1)\, aa_2 = [ga_1 - a_1 g]\, aa_2 = \varphi(a_1 aa_2) - a_1 \varphi(a)\, a_2$ and so γ is inner.

Now assume that A is $(A_+ - A_+, \mathbb{C} - A_+)$-amenable. Since $A \otimes_h A$ is in ${}_{A_+}O_{A_+}$ we have by 7.3 that $(A \otimes_h A)^{**}$ is in ${}_{A+}O_{A+}$. Let $\gamma: A_+ \times A \times A_+ \to (A \otimes_h A)^{**}$ be defined by $\gamma(a_1, a, a_2) = a_1 \otimes aa_2 - 1 \otimes a_1 aa_2$. Here we use the fact that $A \subseteq A^{**}$ and $A_+ \otimes_h A_+ \subseteq A^{**} \otimes_h A^{**} \subseteq (A \otimes_h A)^{**}$ with all containments completely isometric. Let $\pi: A \otimes_h A \to A$ be the product map and let $K = \ker \pi^{**} = [(A \otimes_h A)^*/(\operatorname{Im} \pi^*)^-]^*$ so that K is a completely bounded dual $A_+ - A_+$-bimodule. Now $\pi^{**}(\gamma(a_1, a, a_2)) = \pi_+(\gamma(a_1, a, a_2)) = 0$ where $\pi_+: A_+ \otimes_h A_+ \to A_+$ is the product map. Hence γ is actually into K. Since $E_{cb}xt^1(A_+ - A_+, \mathbb{C} - A_+; A, K) = 0$, γ is necessarily inner. Thus, there exists $\varphi: A \to K$ with $a_1 \otimes aa_2 - 1 \otimes a_1 aa_2 = a_1 \varphi(a)\, a_2 - \varphi(a_1 aa_2)$. Let $\{e_\alpha\}$ be a 2-sided bounded approximate identity for A and let W be a weak*-limit point of $\varphi(e_\alpha)$. Then

$$
\begin{aligned}
a \cdot W - W \cdot a &= \lim_\alpha a\varphi(e_\alpha) - \varphi(e_\alpha)\, a \\
&= \lim_\alpha a\varphi(e_\alpha) \cdot 1 - \varphi(ae_\alpha 1) + \varphi(ae_\alpha - e_\alpha a) \\
&\quad + \varphi(1 \cdot e_\alpha \cdot a) - 1 \cdot \varphi(e_\alpha)\, a \\
&= \lim_\alpha a \otimes e_\alpha - 1 \otimes ae_\alpha + \varphi(ae_\alpha - e_\alpha a) \\
&= a \otimes 1 - 1 \otimes a = a(1 \otimes 1) - (1 \otimes 1) \cdot a.
\end{aligned}
$$

Hence if we set $v = 1 \otimes 1 - W$ then $v \in (A \otimes_h A)^{**}$, $a \cdot v - v \cdot a = 0$ and $\pi^{**}(v) = \pi^{**}(1 \otimes 1) = 1$. ∎

We are now in a position to characterize the C^*-algebras which satisfy our notion of amenability.

THEOREM 7.5. *Let A be a not necessarily unital C^* -algebra. Then A is $(A_+ - A_+, \mathbb{C} - A_+)$-amenable if and only if A is nuclear.*

Proof. Let $\rho: A \to B(H)$ be a non-degenerate *-homomorphism, and let $\{\mu_\alpha\}$ be an approximate diagonal, $\mu_\alpha = \sum x_n^\alpha \otimes y_n^\alpha$. Define $\Phi_\alpha: B(H) \to B(H)$ as before $\Phi_\alpha(T) = \sum \rho(x_n^\alpha)\, T\rho(y_n^\alpha)$, then $\{\Phi_\alpha\}$ is a net of completely bounded maps whose *cb*-norms are uniformly bounded. Using the compactness of balls in the BW-topology [Pa1] we may choose a BW-limit point Φ and a subnet $\{\Phi_{\alpha_\beta}\}$ such that $\langle \Phi(T)\, h, k \rangle = \lim_\beta \langle \Phi_{\alpha_\beta}(T)\, h, k \rangle$ for all vectors h, k in H. It is easily checked that the properties of an approximate diagonal guarantees that the range of Φ is contained in $\rho(A)'$, $\Phi(I) = I$, and Φ is a $\rho(A)'$-bimodule map. Thus, Φ is a quasi-expectation and so by [BuPa], we have that $\rho(A)'$ is an injective von

Neumann algebra. Consequently, $\rho(A)''$ is injective and since this is true for every ρ, by [CE1] and [CE2], A is nuclear.

Conversely, if A is nuclear, then by [Ha1] A is amenable as a Banach algebra which by [Jo] implies that A has an approximate diagonal in the projective tensor product of A with itself. Since the projective tensor product is a larger norm than the Haagerup tensor product, the image of this approximate diagonal defines the desired approximate diagonal. ∎

We close this section with an example of an amenable module motivated by operator theory. Let U be a unitary operator on a Hilbert space H, and regard H_c as a completely contractive left $A(\mathbb{D})$-module, so that H_c is in ${}_{A(\mathbb{D})}O_{\mathbb{C}}$.

Proposition 7.6. *Let U be a unitary operator on a Hilbert space H, then H_c is $(A(\mathbb{D})-\mathbb{C}, \mathbb{C}-\mathbb{C})$-amenable.*

Proof. Let $X=Y^*$ be a completely contractive dual left $A(\mathbb{D})$-module and let $\gamma\colon A(\mathbb{D})\times H_c\to X$ be a completely bounded $A(\mathbb{D})-\mathbb{C}$-derivation. Define $\varphi_n\colon H_c\to X$ via $\varphi_n(h)=\gamma(z^n, U^{-n}h)$ then it is readily checked that each $\|\varphi_n\|_{cb}\leqslant\|\gamma\|_{cb}$. Fix a generalized limit, GLIM and define $\varphi\colon H_c\to X$ via

$$\varphi(h)(y)=GLIM\{\gamma(z^n, U^{-n}h)(y)\}$$

Then φ is completely bounded with $\|\varphi\|_{cb}\leqslant\|\gamma\|_{cb}$ and we claim that γ is the inner derivation implemented by φ. To see this first check that for any $k\geqslant 0$,

$$\begin{aligned} z^k\varphi(h)(y) &= GLIM\{z^k\gamma(z^n, U^{-n}h)(y)\} \\ &= GLIM\{\gamma(z^{n+k}, U^{-n}h)(y)-\gamma(z^k, z^n\cdot U^{-n}h)(y)\} \\ &= GLIM\{\gamma(z^{n+k}, U^{-(n+k)}U^kh)(y)\}-\gamma(z^k, h)(y) \\ &= \varphi(U^kh)(y)-\gamma(z^k, h)(y). \end{aligned}$$

Hence, $\varphi(z^kh)-z^k\varphi(h)=\varphi(U^kh)-\varphi(U^kh)+\gamma(z^k, h)=\gamma(z^k, h)$. From this it follows that φ implements γ, by using the density of the polynomials in $A(\mathbb{D})$. ∎

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